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Weighted Poincaré inequalities

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Poincaré-type inequalities are a key tool in the analysis of partial differential equations. They play a particularly central role in the analysis of domain decomposition and multilevel iterative methods for second-order elliptic problems. When the diffusion coefficient varies within a subdomain or within a coarse grid element, then condition number bounds for these methods based on standard Poincaré inequalities may be overly pessimistic. In this paper, we present new results on weighted Poincaré-type inequalities for very general classes of coefficients that lead to sharper bounds independent of any possible large variation in the coefficients. The main requirement on the coefficients is some form of quasi-monotonicity that we will carefully describe and analyse. The Poincaré constants depend on the topology and the geometry of regions of relatively high and/or low coefficient values, and we shall study these dependencies in detail. Applications of the inequalities in the analysis of domain decomposition and multigrid methods can be found in Pechstein & Scheichl (2011, Numer. Math., 118) and Scheichl et al. (2012, SIAM J. Numer. Anal., 50).

Keywords: Poincare/Friedrichs inequalities; discrete Sobolev inequalities; quasi-monotone heterogeneous coefficients; coefficient robustness; geometric dependencies.

1. Introduction

Poincaré-type inequalities are a key tool in the analysis of partial differential equations (PDEs). They are at the heart of uniqueness results, of *a priori* and *a posteriori* error analyses of discretization schemes, and of convergence analyses of iterative solution strategies, in particular, in the analysis of domain decomposition (DD) and multigrid (MG) methods for finite element (FE) discretizations of elliptic PDEs of the type

$$-\nabla \cdot (\alpha \nabla u) = f. \tag{1.1}$$

In many applications, such as porous media flow or electrostatics, the coefficient function $\alpha = \alpha(x)$ in (1.1) is discontinuous and varies over several orders of magnitude throughout the domain in a possibly very complicated way. Standard analyses of multilevel iterative methods for (1.1) that use classical Poincaré-type inequalities will often lead to pessimistic bounds in this case. If the subdomain partition in a DD method or the coarsest grid in an MG method can be chosen such that $\alpha(x)$ is constant (or almost constant) on each subdomain or on each coarse grid element, then it is possible to prove bounds that are independent of the coefficient variation (cf. Dryja *et al.*, 1996; Klawonn & Widlund, 2001; Toselli & Widlund, 2005; Xu & Zhu, 2008). However, if this is not possible and the coefficient varies strongly within a subdomain or within a coarse grid element, then the classical bounds depend on the local variation of the coefficient, which may be overly pessimistic in many cases. To obtain sharper bounds in some of these cases, it is possible to refine the standard analyses and use Poincaré inequalities on annulus-type boundary layers of each subdomain (Graham *et al.*, 2007; Scheichl & Vainikko, 2007; Pechstein & Scheichl, 2008, 2009) or weighted Poincaré-type inequalities (Galvis & Efendiev, 2010a; Pechstein & Scheichl, 2011; Scheichl *et al.*, 2012); see also Sarkis (1997), Oswald (1999), Griebel *et al.* (2007), Dohrmann *et al.* (2008), Klawonn *et al.* (2008), Zhu (2008), Efendiev & Galvis (2010) and Galvis & Efendiev (2010b) for related work.

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. Throughout the paper, we consider coefficients or weight functions α with

$$\alpha \in L^\infty_+(D) := \left\{ \alpha \in L^\infty(D) : \inf_{x\in D} \alpha(x) > 0 \right\}. \tag{1.2}$$

Such a weight function induces the weighted norm and seminorm

$$\begin{aligned} \|u\|_{L^2(D),\alpha} &:= \left(\int_D \alpha(x) |u(x)|^2 \, \mathrm{d}x \right)^{1/2}, \\ |u|_{H^1(D),\alpha} &:= \left(\int_D \alpha(x) |\nabla u(x)|^2 \, \mathrm{d}x \right)^{1/2}. \end{aligned} \tag{1.3}$$

Let $C_{\mathrm{P},\alpha}(D)$ be the smallest constant such that the weighted Poincaré-type inequality

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leqslant C_{\mathrm{P},\alpha}(D) \, \mathrm{diam}(D)^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D) \tag{1.4}$$

holds. We are interested in finding bounds for $C_{\mathrm{P},\alpha}(D)$ that are in a certain sense *robust* with respect to strong variations in α. To explain the kind of robustness we strive for, we give a preview of some of our results. If α is piecewise constant with respect to a partition of D, we will be able to show that under a suitable monotonicity condition on α, there exists a bound for $C_{\mathrm{P},\alpha}(D)$ that is independent of the *values* of α. Furthermore, we can make explicit the dependence on the geometry of the partition. Bounds for coefficients α that are not piecewise constant can easily be deduced from these results. However, they depend on some local variations of α.

Clearly, $C_{\mathrm{P},\alpha}(D)$ depends on the shape of the domain D. However, one easily shows by dilation that $C_{\mathrm{P},\alpha}(D)$ is independent of $\mathrm{diam}(D)$. The infimum in (1.4) is attained when choosing the constant

$$c = \bar{u}^{D,\alpha} := \frac{\int_D \alpha u \, \mathrm{d}x}{\int_D \alpha \, \mathrm{d}x}, \tag{1.5}$$

which is the α-weighted average of u over D (cf., for example, Chua & Wheeden, 2006; Galvis & Efendiev, 2010a, Lemma 4). This is easily seen from a variational argument. The functional on the left-hand side of (1.4) is convex with respect to c, and hence the infimum is attained if and only if

$$0 = \frac{\mathrm{d}}{\mathrm{d}c} \int_D \alpha |u - c|^2 \, \mathrm{d}x = -2 \int_D \alpha (u - c) \, \mathrm{d}x.$$

If $\mathrm{diam}(D) = 1$, the best constant $C_{\mathrm{P},\alpha}(D)$ is the inverse of the second smallest eigenvalue of the generalized eigenvalue problem

$$-\nabla \cdot (\alpha \nabla u) = \lambda \alpha u \quad \text{in } D, \tag{1.6}$$

$$\alpha \nabla u \cdot n = 0 \quad \text{on } \partial D; \tag{1.7}$$

see, for example, Galvis & Efendiev (2010b). For general weight functions α, we can obtain a bound for $C_{\mathrm{P},\alpha}(D)$ in (1.4) from the usual Poincaré inequality. Let $\bar{u}^D := \bar{u}^{D,1}$ be the usual average (cf. (1.5)). Then, it is easily shown that

$$\|u - \bar{u}^D\|^2_{L^2(D),\alpha} \leqslant \sup_{x,y \in D} \frac{\alpha(x)}{\alpha(y)} C_P(D)\, \mathrm{diam}(D)^2 |u|^2_{H^1(D),\alpha},$$

where $C_P(D) = C_{\mathrm{P},1}(D)$ is the usual Poincaré constant on D. Thus, this bound for $C_{\mathrm{P},\alpha}(D)$ depends on the *global variation* $\sup_{x,y\in D}(\alpha(x)/\alpha(y))$, and if α is highly variable this may be very large and very pessimistic.

We note that although weighted Poincaré inequalities have been investigated a lot in the literature, estimates of the Poincaré constant $C_{\mathrm{P},\alpha}$ that show certain robustness in α are hardly known. Chua (1993) showed that the weighted Poincaré inequality holds for domains satisfying the Boman chain condition with weights α from a Muckenhoupt class (see Muckenhoupt, 1972). Chua's paper is based on the early work by Iwaniec & Nolder (1985); see also Fabes *et al.* (1982) and Maz'ja (1985) for related work. The constant in the Poincaré inequality depends in general on the weight. A similar result was obtained by Zhikov & Pastukhova (2008, Lemma 2.6) for weights $\alpha \in L^r$ with $\alpha^{-1} \in L^s$ with $2d^{-1} = r^{-1} + s^{-1}$. Also there, the Poincaré constant depends on α. Chua & Wheeden (2006) provide explicit estimates for the Poincaré constant for the class of convex domains Ω with weights α that are a positive power of a non-negative concave function. Note that concavity implies *continuity*. Recently, Veeser & Verfürth (2012) refined these results to star-shaped domains, where the weight function satisfies a concavity property with respect to the central point of the star (see Veeser & Verfürth, 2012, Condition (2.3) for more details, and see Veeser & Verfürth, 2009 on how to use these inequalities in (explicit) *a posteriori* error estimation). We also note that Chua (1993), Chua & Wheeden (2006) and Veeser & Verfürth (2012) cover the general case of L^p, not just L^2. To the best of our knowledge, the first paper that deals with robust estimates of the weighted Poincaré constant for *discontinuous* weight functions is Galvis & Efendiev (2010a). There, Galvis and Efendiev show that, for piecewise constant coefficients α, if the largest value is attained in a connected region Ω_1 and if all the other regions of constant α are inclusions of (or at least bordering) Ω_1, then $C_{\mathrm{P},\alpha}$ is independent of the *values* of α, in particular of possibly high *contrast*.

In the present paper, we want to expand on the results in Galvis & Efendiev (2010a) and Pechstein & Scheichl (2011) and present sharp constants for weighted Poincaré-type inequalities that are independent of the range of values of the weight function for a rather general class of coefficients. See also Pechstein & Scheichl (2010) for an advertisement of some preliminary results. In Section 2.1, we will define a class of *quasi-monotone* piecewise constant weight functions (far more general than in Galvis & Efendiev, 2010a) for which we can make $C_{\mathrm{P},\alpha}(D)$ totally independent of the (piecewise constant) *values* of α. To get bounds for $C_{\mathrm{P},\alpha}(D)$ in (1.4), we will choose averages over certain manifolds rather than over D. In Section 2.2, we will achieve similar results for an even more general class of coefficients that are not piecewise constant. In many applications, especially in the analysis of MG and DD methods, Poincaré-type inequalities are not needed on all of $H^1(D)$ but only for the subset of FE functions. This restriction allows for a larger class of coefficients α, where we can show discrete analogues of inequality (1.4). This issue will be treated in Section 3. Even if the Poincaré constant $C_{\mathrm{P},\alpha}(D)$ can be bounded independent of the range of α, it will in general depend on the topology and geometry of the regions where the coefficient α is 'large' and where it is 'small'. To make this more precise, in Section 4, we restrict our attention again to piecewise constant weights and work out what this *geometric* dependence looks like. Since this issue can be rather complicated in two and three space dimensions, we present a series of general technical tools and analyse a few examples in detail.

Extensions to PDEs/inequalities, where α is replaced by an isotropic tensor, are straightforward whereas the case of anisotropic tensors is substantially harder.

Applications of these novel, weighted Poincaré-type inequalities in the analysis of geometric MG, as well as of two-level overlapping Schwarz and FETI DD methods can be found in Pechstein & Scheichl (2011) and Scheichl *et al.* (2012).

2. Weighted Poincaré-type inequalities in H^1

Let us start by considering inequalities for piecewise constant weight functions (Section 2.1). We will return to more general weight functions in Section 2.2.

2.1 *Quasi-monotone piecewise constant weight functions*

Let the weight function $\alpha \in L^\infty_+(D)$ be piecewise constant with respect to a nonoverlapping partitioning of D into open, connected Lipschitz polytopes $\mathcal{Y} := \{Y_\ell : \ell = 1, \ldots, n\}$, that is,

$$\bar{D} = \bigcup_{\ell=1}^{n} \bar{Y}_\ell \quad \text{and} \quad \alpha|_{Y_\ell} \equiv \alpha_\ell \tag{2.1}$$

for some constants α_ℓ. We shall drop this condition in Section 2.2.

To simplify the presentation, we set $H := \mathrm{diam}(D)$ and define, for any $u \in H^1(D)$ and for any $(d-1)$-dimensional manifold $X \subset \bar{D}$, the average

$$\bar{u}^X := \begin{cases} \dfrac{1}{\mathrm{meas}_{d-1}(X)} \displaystyle\int_X u \,\mathrm{d}s & \text{if } d > 1, \\ \dfrac{1}{\mathrm{meas}_{d-1}(X)} \displaystyle\sum_{x \in X} u(x) & \text{if } d = 1, \end{cases}$$

where $\mathrm{meas}_0(X) := \sum_{x \in X} 1$.

DEFINITION 2.1 Suppose that $\alpha \in L^\infty_+(D)$ satisfies (2.1) and let the index $\ell^* \in \{1, \ldots, n\}$ be such that $\alpha_{\ell^*} = \max_{\ell=1,\ldots,n} \alpha_\ell$.

(a) We call the region $P_{\ell_1,\ell_s} := (\bar{Y}_{\ell_1} \cup \bar{Y}_{\ell_2} \cup \cdots \cup \bar{Y}_{\ell_s})^\circ$, $1 \leqslant \ell_1, \ldots, \ell_s \leqslant n$, a *quasi-monotone path* from Y_{ℓ_1} to Y_{ℓ_s} (with respect to α) if the following two conditions are satisfied:

  (i) for each $i = 1, \ldots, s-1$, the regions $\bar{Y}_{\ell_i}$ and $\bar{Y}_{\ell_{i+1}}$ share a common $(d-1)$-dimensional manifold X_i;

  (ii) $\alpha_{\ell_1} \leqslant \alpha_{\ell_2} \leqslant \cdots \leqslant \alpha_{\ell_s}$.

We refer to s as the *length* of P_{ℓ_1,ℓ_s}.

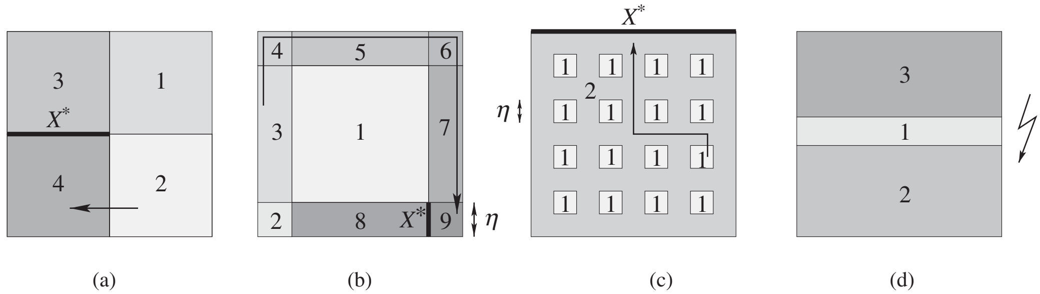


FIG. 1. The numbering of the regions Y_ℓ in these examples is according to the relative sizes of the weights α_ℓ on each region, with the smallest weight in region Y_1. Examples (a,b,c) are quasi-monotone in the sense of Definition 2.1. In each case, a typical path and a suitable manifold X^* are displayed. Example (d) is not quasi-monotone.

(b) We say that α is *quasi-monotone on D* if, for any $k = 1, \ldots, n$, there exists a quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*}. Let s_k denote the length of P_{k,ℓ^*}.

(c) Let $X^* \subset \bar{Y}_{\ell^*}$ be a $(d-1)$-dimensional manifold. For each $k = 1, \ldots, n$, let $c_k^{X^*} > 0$ be the best constant such that

$$\|u - \bar{u}^{X^*}\|^2_{L^2(Y_k)} \leqslant c_k^{X^*} H^2 |u|^2_{H^1(P_{k,\ell^*})} \quad \forall u \in H^1(P_{k,\ell^*}), \tag{2.2}$$

and set $C^*_{\mathrm{P},\alpha} := \sum_{k=1}^n c_k^{X^*}$.

Without loss of generality, we can assume that the index ℓ^* of the subregion with the largest value is unique. If ℓ^* is not unique, then α is either not quasi-monotone, or all the regions where the maximum of α is attained must be connected and can therefore be subsumed into one region Y_{ℓ^*}.

Note that the constant $C^*_{\mathrm{P},\alpha}$ in Definition 2.1(c) depends on the choice of manifold $X^* \subset \bar{Y}_{\ell^*}$ and of the paths $\{P_{k,\ell^*}\}_{k=1}^n$. The above definition is a generalization of the notion of quasi-monotone coefficients introduced in Dryja *et al.* (1996) in the context of DD solvers (see Bernardi & Verfürth, 2000 for an application in *a posteriori* error analysis). In Fig. 1(a,b,c), we give some examples of weight functions that satisfy Definition 2.1. The coefficient shown in Fig. 1(d) fails to be quasi-monotone.

The following theorem shows that, for quasi-monotone weight functions α, the constant $C^*_{\mathrm{P},\alpha}$ from Definition 2.1(c), which is clearly independent of the values that α takes on D, provides a bound on the best constant $C_{\mathrm{P},\alpha}(D)$ in (1.4).

THEOREM 2.2 (Weighted Poincaré inequality—piecewise constant case.) Let $\alpha \in L^\infty_+(D)$ be quasi-monotone on D in the sense of Definition 2.1. Then

$$\inf_{c \in \mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leqslant C^*_{\mathrm{P},\alpha} H^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D), \tag{2.3}$$

where $C^*_{\mathrm{P},\alpha}$ is the constant defined in Definition 2.1(c), that is, $C_{\mathrm{P},\alpha}(D) \leqslant C^*_{\mathrm{P},\alpha}$.

Proof. For simplicity, we assume that $H = \mathrm{diam}(D) = 1$. The general case follows from a dilation argument. We set $c = \bar{u}^{X^*}$ (where X^* is the manifold chosen in Definition 2.1) and assume without loss of generality that $\bar{u}^{X^*} = 0$. Otherwise, we can set $\hat{u} := u - \bar{u}^{X^*}$ and use the fact that $|\hat{u}|_{H^1(D),\alpha} = |u|_{H^1(D),\alpha}$.

Let $k \in \{1, \ldots, n\}$ be fixed. Then, due to the assumption (2.1) on the weight function α, we have

$$\|u\|^2_{L^2(Y_k),\alpha} = \alpha_k \|u\|^2_{L^2(Y_k)}.$$

Combining this identity with inequality (2.2) and using the fact that the value of α is monotonically increasing in the path from Y_k to Y_{ℓ^*}, we obtain

$$\|u\|^2_{L^2(Y_k),\alpha} \leqslant c_k^{X^*} \alpha_k |u|^2_{H^1(P_{k,\ell^*})} \leqslant c_k^{X^*} |u|^2_{H^1(P_{k,\ell^*}),\alpha} \leqslant c_k^{X^*} |u|^2_{H^1(D),\alpha}.$$

The proof is completed by adding up the above estimates for $k = 1, \ldots, n$. □

As we can see from the proof of Theorem 2.2, inequality (2.3) does not only hold for the infimum, that is, for the weighted average $c = \bar{u}^{D,\alpha}$, but also for $c = \bar{u}^{X^*}$ where X^* may be any $(d-1)$-dimensional manifold in Y_{ℓ^*}. This is of importance in certain applications.

Although the definition of the constant $C^*_{\mathrm{P},\alpha}$ in Definition 2.1(c) suggests that it grows with the number n of subregions, this is not the case in general. The reason is that on the left-hand side in (2.2), the L^2 norm is taken only over Y_k and *not* over the whole path P_{k,ℓ^*}. We will discuss this issue extensively in Section 4. However, we would like to give already at this stage a general tool, Lemma 2.4, on how the inequalities (2.2) are related to more common Poincaré inequalities on each of the individual subregions Y_k. We note that a similar decomposition procedure can already be found in Veeser & Verfürth (2012, Section 2.3).

DEFINITION 2.3 For any bounded Lipschitz domain $Y \subset \mathbb{R}^d$ and for any $(d-1)$-dimensional manifold $X \subset \bar{Y}$, let $C_P(Y; X) > 0$ denote the best constant such that the following Poincaré-type inequality holds:

$$\|u - \bar{u}^X\|^2_{L^2(Y)} \leqslant C_P(Y; X)\, \mathrm{diam}(Y)^2 |u|^2_{H^1(Y)} \quad \forall u \in H^1(Y). \tag{2.4}$$

LEMMA 2.4 Suppose $\alpha \in L^\infty_+(D)$ is quasi-monotone and P_{k,ℓ^*} is any of the paths in Definition 2.1(b) with $\ell_1 = k$ and $\ell_s = \ell^*$. For convenience let $X_0 := X_1$ and $X_s := X^*$. Then the constant $c_k^{X^*}$ in Definition 2.1(c) can be bounded by

$$c_k^{X^*} \leqslant 4 \sum_{i=1}^{s} \frac{\mathrm{meas}(Y_k)}{\mathrm{meas}(Y_{\ell_i})} \frac{\mathrm{diam}(Y_{\ell_i})^2}{H^2} \max\{C_P(Y_{\ell_i}; X_{i-1}), C_P(Y_{\ell_i}; X_i)\}.$$

Proof. By a telescoping argument we have

$$\|u - \bar{u}^{X^*}\|_{L^2(Y_k)} \leqslant \|u - \bar{u}^{X_1}\|_{L^2(Y_k)} + \sum_{i=2}^{s} \sqrt{\mathrm{meas}(Y_k)} |\bar{u}^{X_{i-1}} - \bar{u}^{X_i}|. \tag{2.5}$$

Estimate (2.4) yields a bound for the first term on the right-hand side, that is,

$$\|u - \bar{u}^{X_1}\|^2_{L^2(Y_k)} \leqslant C_P(Y_k; X_1)\, \mathrm{diam}(Y_k)^2 |u|^2_{H^1(Y_k)}. \tag{2.6}$$

For i fixed, we can also conclude from inequality (2.4) that

$$|\bar{u}^{X_{i-1}} - \bar{u}^{X_i}|^2 \leqslant \frac{2}{\mathrm{meas}(Y_{\ell_i})}\left(\|\bar{u}^{X_{i-1}} - u\|^2_{L^2(Y_{\ell_i})} + \|u - \bar{u}^{X_i}\|^2_{L^2(Y_{\ell_i})}\right)$$
$$\leqslant 4\max\{C_P(Y_{\ell_i};X_{i-1}), C_P(Y_{\ell_i};X_i)\}\frac{\mathrm{diam}(Y_{\ell_i})^2}{\mathrm{meas}(Y_{\ell_i})}|u|^2_{H^1(Y_{\ell_i})} \tag{2.7}$$

(this is essentially a Bramble–Hilbert-type argument). An application of Cauchy's inequality (in $\mathbb{R}^s$) yields the final result. □

Note that in one dimension, due to Lemma 2.4, the Poincaré constant $C^*_{\mathrm{P},\alpha}$ is $\mathcal{O}(1)$ as $n \to \infty$, as the following corollary shows. The situation in higher dimensions is more complicated and is left until Section 4.

COROLLARY 2.5 Let $d = 1$. If α is piecewise constant with respect to $\{Y_\ell\}_{\ell=1}^n$ and quasi-monotone in the sense of Definition 2.1, then $C^*_{\mathrm{P},\alpha} = \mathcal{O}(1)$ as $n \to \infty$.

Proof. We assume without loss of generality, that $D = (0,1)$ and $X^* = 1$. (Note that in this case quasi-monotonicity in the sense of Definition 2.1 is equivalent to the usual monotonicity.) Let us assume that the regions Y_ℓ are numbered consecutively from left to right, and that $X_\ell := \bar{Y}_\ell \cap \bar{Y}_{\ell+1}$ for $\ell = 1,\dots,n-1$, with $X_n := X^*$. It follows from the Fundamental Theorem of Calculus that

$$\|u - u(X_{\ell-1})\|^2_{L^2(Y_\ell)} \leqslant \mathrm{diam}(Y_\ell)^2|u|^2_{H^1(Y_\ell)} \quad \forall u \in H^1(Y_\ell),\ \ \forall \ell = 1,\dots,n. \tag{2.8}$$

Hence, $C_P(Y_\ell;X_{\ell-1}) \leqslant 1$. The same is true if we replace $X_{\ell-1}$ by X_ℓ. Since, for $d = 1$, we have $\mathrm{meas}(Y_\ell) = \mathrm{diam}(Y_\ell)$, it follows from Lemma 2.4 that

$$c_k^{X^*} \leqslant 4\,\mathrm{diam}(Y_k)\sum_{\ell=k}^{n}\mathrm{diam}(Y_\ell) \leqslant 4\,\mathrm{diam}(Y_k) \quad \forall k = 1,\dots,n,$$

and so $C^*_{\mathrm{P},\alpha} \leqslant 4 = \mathcal{O}(1)$ as $n \to \infty$. □

Note that it was crucial to define $c_k^{X^*}$ as done in Definition 2.1. Using a standard Poincaré-type inequality for P_{k,ℓ^*}, such as

$$\|u - \bar{u}^{X^*}\|^2_{L^2(P_{k,\ell^*})} \leqslant C_P(P_{k,\ell^*};X^*)\,\mathrm{diam}(P_{k,\ell^*})^2|u|^2_{H^1(P_{k,\ell^*})},$$

would lead to a very pessimistic bound for the Poincaré constant in (2.3):

$$C^*_{\mathrm{P},\alpha} \leqslant \sum_{k=1}^{n} C_P(P_{k,\ell^*};X^*)\frac{\mathrm{diam}(P_{k,\ell^*})^2}{H^2}.$$

In our one-dimensional example in Corollary 2.5 this would in general lead to $C^*_{\mathrm{P},\alpha} = \mathcal{O}(n)$.

An inequality similar to that in Theorem 2.2 holds if u vanishes on part of the boundary of D. This is sometimes referred to as a Friedrichs inequality.

DEFINITION 2.6 Suppose $\alpha \in L^\infty_+(D)$ satisfies (2.1) and $\Gamma \subset \partial D$.

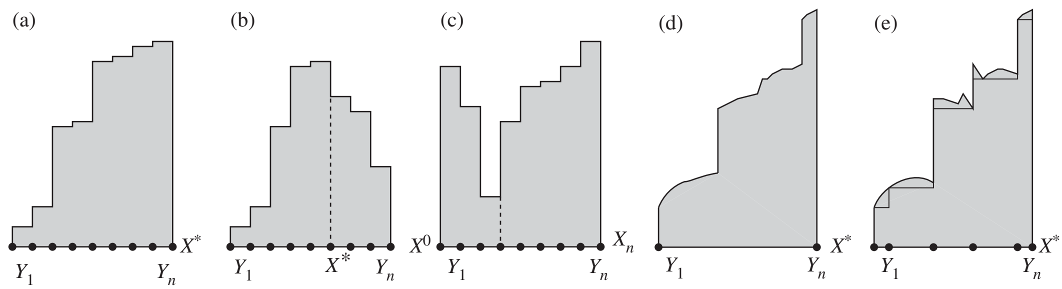


FIG. 2. Examples of quasi-monotone weight functions in one dimension. Cases (a,b) are quasi-monotone in the sense of Definition 2.1. Case (c) is Γ-quasi-monotone in the sense of Definition 2.6 with $\Gamma = \{X_0, X_n\}$, cases (d,e) are quasi-monotone in the sense of Definition 2.8 (see Section 2.2).

(a) We say that α is *Γ-quasi-monotone on D* if, $\forall\, k = 1, \dots, n$, there exist an index ℓ_k^* and a quasi-monotone path P_{k,ℓ_k^*} (with respect to α) from Y_k to $Y_{\ell_k^*}$, such that $\partial Y_{\ell_k^*} \cap \Gamma$ is a $(d-1)$-dimensional manifold.

(b) For each $k = 1, \dots, n$, let $c_k^\Gamma > 0$ be the best constant such that

$$\|u\|^2_{L^2(Y_k)} \leqslant c_k^\Gamma H^2 |u|^2_{H^1(P_{k,\ell_k^*})} \quad \forall u \in H^1(P_{k,\ell_k^*}),\ \ u|_\Gamma = 0, \tag{2.9}$$

and set $C^\Gamma_{\mathrm{F},\alpha} := \sum_{k=1}^n c_k^\Gamma$.

Again the constant $C^\Gamma_{\mathrm{F},\alpha}$ in Definition 2.6(b) is clearly independent of the actual values that α takes on D. A one-dimensional example of a Γ-quasi-monotone function is given in Fig. 2(c). Note that this function is not quasi-monotone in the sense of Definition 2.1, while the example in Fig. 2(b) is not Γ-quasi-monotone in the sense of Definition 2.6 for any choice of $\Gamma \subset \partial D$.

THEOREM 2.7 (Weighted Friedrichs inequality—piecewise constant case.) Let $\Gamma \subset \partial D$ and suppose that $\alpha \in L^\infty_+(D)$ is Γ-quasi-monotone on D in the sense of Definition 2.6. Then

$$\|u\|^2_{L^2(D),\alpha} \leqslant C^\Gamma_{\mathrm{F},\alpha} H^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D) \text{ with } u|_\Gamma = 0,$$

where $C^\Gamma_{\mathrm{F},\alpha}$ is the constant defined in Definition 2.6(b).

Proof. The proof is analogous to that of Theorem 2.2. □

For the remainder of this paper we will restrict our attention to weighted Poincaré-type inequalities (cf. Theorem 2.2), but we remark that there are always analogous statements for weighted Friedrichs-type inequalities (cf. Theorem 2.7) that we will not mention or prove explicitly.

2.2 *General weight functions*

In this subsection, we digress briefly to discuss more general nonconstant weight functions. To do this, we generalize our definition of quasi-monotonicity. Our bounds are then not completely independent of the values of α, but they will only depend on the *local variation*. Finally, we will show that our bounds are in a certain sense sharp.

DEFINITION 2.8 Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be a nonoverlapping partition of D. A weight function $\alpha \in L^\infty_+(D)$ is called (*macroscopically*) *quasi-monotone* with respect to $\mathcal{Y}$ if the auxiliary, piecewise constant weight function $\underline{\alpha} \in L^\infty_+(D)$ defined by

$$\underline{\alpha}(x) := \inf_{y \in Y_\ell} \alpha(y) \quad \forall x \in Y_\ell,$$

is quasi-monotone on D in the sense of Definition 2.1. (For a typical example see Fig. 2(e).)

Clearly, Definition 2.8 is a generalization of Definition 2.1. Any $\alpha \in L^\infty_+(D)$ that satisfies (2.1) and is quasi-monotone in the sense of Definition 2.1 is also macroscopically quasi-monotone with respect to $\mathcal{Y}$ in the sense of Definition 2.8 with $\underline{\alpha} \equiv \alpha$. Moreover, any weight function $\alpha \in L^\infty_+(D)$ is macroscopically quasi-monotone in the sense of Definition 2.8 with respect to the trivial partition $\mathcal{Y} := \{D\}$. However, a finer partition may lead to a better bound for the Poincaré constant $C_{\mathrm{P},\alpha}(D)$ in the following theorem (which is a generalization of Theorem 2.2).

Analogously to $\underline{\alpha}$ let us also define $\bar{\alpha} \in L^\infty_+(D)$ such that

$$\bar{\alpha}(x) := \sup_{y \in Y_\ell} \alpha(y) \quad \forall x \in Y_\ell.$$

THEOREM 2.9 (Weighted Poincaré inequality—general case.) Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be a nonoverlapping partition of D and let $\alpha \in L^\infty_+(D)$ be macroscopically quasi-monotone with respect to $\mathcal{Y}$ in the sense of Definition 2.8. Then

$$\inf_{c \in \mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leqslant C^*_{\mathrm{P},\underline{\alpha}} \left\| \frac{\bar{\alpha}}{\underline{\alpha}} \right\|_{L^\infty(D)} H^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D),$$

where $C^*_{\mathrm{P},\underline{\alpha}}$ is the constant in Definition 2.1(c) for the auxiliary function $\underline{\alpha}$.

Proof. We proceed as in the proof of Theorem 2.2 and assume without loss of generality that $\bar{u}^{X^*} = 0$ and $\mathrm{diam}(D) = 1$. Then, again using Theorem 2.2, inequality (2.2) and the quasi-monotonicity of $\underline{\alpha}$, we have

$$\begin{aligned}\|u\|^2_{L^2(Y_k),\alpha} &\leqslant \sup_{x \in Y_k} \alpha(x) \|u\|^2_{L^2(Y_k)} \\ &\leqslant \sup_{x \in Y_k} \alpha(x) c_k^{X^*} |u|^2_{H^1(P_{k,\ell^*})} \leqslant \frac{\sup_{x \in Y_k} \alpha(x)}{\inf_{y \in Y_k} \alpha(y)} c_k^{X^*} |u|^2_{H^1(P_{k,\ell^*}),\underline{\alpha}}.\end{aligned}$$

Obviously, $|u|_{H^1(P_{k,\ell^*}),\underline{\alpha}} \leqslant |u|_{H^1(P_{k,\ell^*}),\alpha}$, which completes the proof. □

Theorem 2.9 implies the bound $C_{\mathrm{P},\alpha}(D) \leqslant C^*_{\mathrm{P},\underline{\alpha}} \|\bar{\alpha}/\underline{\alpha}\|_{L^\infty(D)}$ for the weighted Poincaré constant in (1.4). This bound is independent of the values of $\underline{\alpha}$ but it depends on the *local* variation of α on each of the subregions $Y_k \in \mathcal{Y}$. However, since we are free to choose the partition $\mathcal{Y}$, it is in principle possible to obtain a Poincaré constant that is completely independent of the variation of α (even for exponentially growing coefficients) by letting $n \to \infty$—provided α remains macroscopically quasi-monotone with respect to $\mathcal{Y}$ as we let $n \to \infty$. We would like to illustrate this in one dimension. The following corollary follows immediately from Theorem 2.9 and the proof of Corollary 2.5.

COROLLARY 2.10 Let $D=[0,1]$ and $X^*\in[0,1]$. If α is monotonically nondecreasing on $(0,X^*)$ and monotonically nonincreasing on $(X^*,1)$, then

$$\inf_{c\in\mathbb{R}}\|u-c\|^2_{L^2(D),\alpha}\leqslant 4|u|^2_{H^1(D),\alpha}\quad \forall u\in H^1(D).$$

Theorem 2.9 also shows that we still get good bounds for $C_{\mathrm{P},\alpha}(D)$, even if we do not have *strict* quasi-monotonicity (in the sense of Definition 2.1). An example of this is the case in Fig. 1(d) with $\alpha_1=1$, $\alpha_2=10$ and $\alpha_3\gg 10$. Applying Theorem 2.9 with the partition $\mathcal{Y}:=\{Y_1\cup Y_2, Y_3\}$ (instead of Theorem 2.2), the maximum local variation is $\|\bar{\alpha}/\underline{\alpha}\|_{L^\infty(D)}=10$ and so it follows from Theorem 2.9 that $C_{\mathrm{P},\alpha}(D)=\mathcal{O}(1)$ as $\alpha_3\to\infty$.

However, the bound in Theorem 2.9 deteriorates when quasi-monotonicity is strongly violated. For the example in Fig. 1(d) it can be shown that

$$C_{\mathrm{P},\alpha}(D)\geqslant c\min\left\{\frac{\alpha_2}{\alpha_1},\frac{\alpha_3}{\alpha_1}\right\}$$

(cf. Pechstein & Scheichl, 2011, Section 3.3). Lemma 3.7 in Section 3.1 shows that quasi-monotonicity is in fact a necessary condition for $C_{\mathrm{P},\alpha}(D)$ to remain bounded when the contrast in the coefficient goes to infinity.

3. Weighted Poincaré inequalities for FE functions

In many applications, for example, in the analysis of multilevel iterative methods for (1.1), it is sufficient to have Poincaré-type inequalities for FE functions. We will show now that it is possible to extend the class of weight functions α for which we can obtain weighted Poincaré inequalities to include piecewise constant functions α that clearly fall outside the original definition of quasi-monotonicity in Dryja *et al.* (1996) and that of the previous section.

Hence, for this section let D be a Lipschitz polytope domain in $\mathbb{R}^d$ ($d\geqslant 2$). For a suitable index set Θ, let $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ be a family of shape-regular simplicial triangulations, that is, there exists a uniform constant $c_{\mathrm{reg}}>0$ such that, $\forall\, h\in\Theta$ and $\forall\,\tau\in\mathcal{T}_h(D)$,

$$\frac{\mathrm{diam}(\tau)}{\rho(\tau)}\leqslant c_{\mathrm{reg}},\tag{3.1}$$

where $\rho(\tau)$ is the diameter of the largest inscribed ball (cf. Ciarlet, 2002). For each $h\in\Theta$, we define the usual space of continuous, piecewise linear FEs

$$V_h(D):=\{v\in\mathcal{C}(\bar{D}): v_{|\tau}\text{ affine linear }\forall\tau\in\mathcal{T}_h(D)\}.$$

Let $\alpha\in L^\infty_+(D)$ be piecewise constant again with respect to a nonoverlapping partitioning of D into open, connected Lipschitz polytopes $\mathcal{Y}:=\{Y_\ell:\ell=1,\ldots,n\}$ such that

$$\bar{D}=\bigcup_{\ell=1}^{n}\bar{Y}_\ell\quad\text{and}\quad \alpha|_{Y_\ell}\equiv\alpha_\ell\tag{3.2}$$

for some constants α_ℓ. In addition, we assume here that α is piecewise constant with respect to $\mathcal{T}_h(D)$, so that $\mathcal{T}_h(D)$ is aligned with $\mathcal{Y}$.

The following lemma is the crucial tool to extend our results to more general coefficients in the case of FE functions. It requires in addition that, restricted to a subregion Y_ℓ, the family $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ is quasi-uniform, that is, there exists a uniform constant $c_{\text{quasi}} > 0$ such that, $\forall\, h \in \Theta$ and $\forall\, \tau, \tau' \in \mathcal{T}_h(Y_\ell)$,

$$\frac{\operatorname{diam}(\tau)}{\operatorname{diam}(\tau')} \leqslant c_{\text{quasi}}. \tag{3.3}$$

Furthermore, we need to define the indicator function

$$\sigma^j(x) := \begin{cases} 1 & \text{if } j = 1, \\ 1 + \log(x) & \text{if } j = 2, \\ x^{j-2} & \text{if } j \geqslant 3. \end{cases} \tag{3.4}$$

LEMMA 3.1 Let Y be a nondegenerate d-dimensional simplex or a d-dimensional hypercube and let $\{\mathcal{T}_h(Y)\}_{h\in\Theta}$ be a quasi-uniform family of simplicial triangulations. Suppose that $X \in \partial Y$ is an m-dimensional facet (vertex, edge, face, etc.) with $0 \leqslant m \leqslant d-1$. Then there exists a constant c_{discr} independent of h, $H = \operatorname{diam}(Y)$, and X such that, $\forall h \in \Theta$ and $\forall\, u \in V_h(Y)$,

$$\|u - \bar{u}^X\|^2_{L^2(Y)} \leqslant c_{\text{discr}} \sigma^{d-m}\left(\frac{H}{h}\right) H^2 |u|^2_{H^1(Y)}.$$

The constant c_{discr} depends on d, m, the ratio $\operatorname{diam}(Y)/\rho(Y)$ and on the constants c_{reg} and c_{quasi} in (3.1) and (3.3).

Proof. Proofs for $d \leqslant 3$ can be found in Toselli & Widlund (2005, Section 4.6); see also Bramble & Xu (1991). A proof for arbitrary dimension is given in Appendix A. □

Note that, clearly, the dependence of the Poincaré constant on H/h gets weaker as the dimension m of the manifold over which we 'average' the function increases. It is linear if $m = d-3$ (for example, $d = 3$ and X is a vertex), logarithmic if $m = d-2$ (for example, $d = 2$ and X is a vertex or $d = 3$ and X is an edge) and it does not depend on H/h at all if $m = d-1$.

DEFINITION 3.2 Suppose that $\alpha \in L^\infty_+(D)$ satisfies (3.2) and that $\ell^* \in \{1,\ldots,n\}$ is such that $\alpha_{\ell^*} = \max_{\ell=1,\ldots,n} \alpha_\ell$. Furthermore, let m be an integer between 0 and $d-1$.

(a) We call the region $P_{\ell_1,\ell_s} := (\bar{Y}_{\ell_1} \cup \bar{Y}_{\ell_2} \cup \cdots \cup \bar{Y}_{\ell_s})^\circ$, $1 \leqslant \ell_1, \ldots, \ell_s \leqslant n$, a *type-m quasi-monotone path* from Y_{ℓ_1} to Y_{ℓ_s} (with respect to α) if the following two conditions hold:

(i) for each $i = 1, \ldots, s-1$, the regions $\bar{Y}_{\ell_i}$ and $\bar{Y}_{\ell_{i+1}}$ share a common m-dimensional manifold X_i;

(ii) $\alpha_{\ell_1} \leqslant \alpha_{\ell_2} \leqslant \cdots \leqslant \alpha_{\ell_s}$.

(b) We say that α is *type-m quasi-monotone on D* if, $\forall\, k = 1, \ldots, n$, there exists a quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*}. Again, s_k denotes the length of P_{k,ℓ^*}.

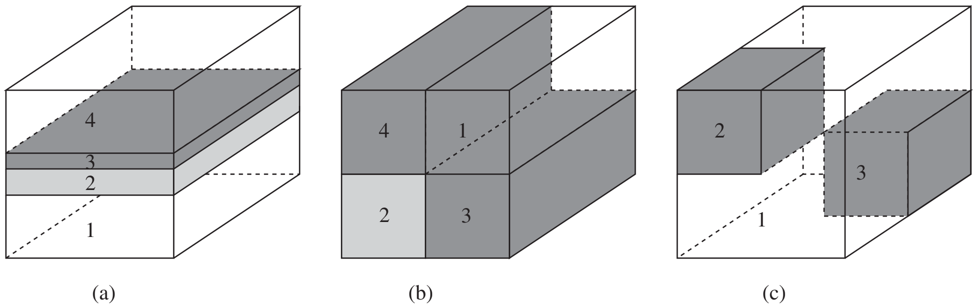


FIG. 3. Examples of type-m quasi-monotone weight functions for $d = 3$ with $m \leqslant 2$ in (a), with $m \leqslant 1$ in (b) and with $m = 0$ in (c).

(c) Let $X^* \subset \bar{Y}_{\ell^*}$ be an m-dimensional manifold and for each $k = 1, \ldots, n$ let $c_k^{X^*} > 0$ be the best constant such that, $\forall\, h \in \Theta$,

$$\|u - \bar{u}^{X^*}\|^2_{L^2(Y_k)} \leqslant c_k^{X^*} \sigma^{d-m}\left(\frac{H}{h}\right) H^2 |u|^2_{H^1(P_{k,\ell^*})} \quad \forall u \in V_h(P_{k,\ell^*}). \tag{3.5}$$

As before, we set $C^*_{\mathrm{P},\alpha} := \sum_{k=1}^n c_k^{X^*}$.

Clearly, a type-m quasi-monotone coefficient α is also type-$(m-1)$ quasi-monotone. In Fig. 3, we see some examples. The examples in Fig. 3(b,c) are clearly not quasi-monotone in the classical sense (cf. Dryja *et al.*, 1996), yet a discrete version of the weighted Poincaré inequality in Theorem 2.2 can be established even for these coefficients, with a constant that does not depend on α.

REMARK 3.3 (a) If the index ℓ^* is not unique, then α is either not type-m quasi-monotone, or there exists a type-m quasi-monotone path connecting all the regions where the maximum of α is attained. The union of these regions is then called *type-m connected*.

(b) Note that without any additional work, the following theory can in fact be extended to regions $Y_1, \ldots, Y_n$ that are only type-m connected rather than connected. To simplify the presentation, we chose not to do so.

THEOREM 3.4 (Discrete weighted Poincaré inequality) Let $0 \leqslant m \leqslant d-1$ and let $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ be quasi-uniform. If $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone on D in the sense of Definition 3.2, then

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leqslant C^*_{\mathrm{P},\alpha} \sigma^{d-m}\left(\frac{H}{h}\right) H^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in V_h(D), \tag{3.6}$$

where $C^*_{\mathrm{P},\alpha}$ is as in Definition 3.2(c) and $\sigma^{d-m}(H/h)$ is defined in (3.4).

Proof. The proof is identical to the proof of Theorem 2.2 using (3.5) instead of (2.2). □

Let us finish this section by analysing again how the inequality (3.5) is related to inequalities on the individual subregions Y_k.

DEFINITION 3.5 For any bounded Lipschitz domain $Y \subset D$ resolved by $\mathcal{T}_h(D)$, and for any m-dimensional manifold $X \subset \bar{Y}$, let $C_P(Y; X) > 0$ denote the best constant such that, $\forall\, h \in \Theta$

and $\forall\, u \in V_h(Y)$,

$$\|u-\bar{u}^X\|^2_{L^2(Y)} \leqslant C_P(Y;X)\sigma^{d-m}\left(\frac{\mathrm{diam}(Y)}{h}\right)\mathrm{diam}(Y)^2|u|^2_{H^1(Y)}. \tag{3.7}$$

The existence of such a constant $C_P(Y;X)$ is guaranteed, for example, in the case of a simplex or hypercube if X is one of the facets of Y (cf. Lemma 3.1).

LEMMA 3.6 Suppose $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone and P_{k,ℓ^*} is any of the paths in Definition 3.2(b) with $\ell_1 = k$ and $\ell_s = \ell^*$. For convenience let $X_0 := X_1$ and $X_s := X^*$. Then the constant $c_k^{X^*}$ in Definition 3.2(c) can be bounded by

$$c_k^{X^*} \leqslant 4\sum_{i=1}^{s}\frac{\mathrm{meas}(Y_k)}{\mathrm{meas}(Y_{\ell_i})}\frac{\mathrm{diam}(Y_{\ell_i})^2}{H^2}\max\{C_P(Y_{\ell_i};X_{i-1}), C_P(Y_{\ell_i};X_i)\}.$$

Proof. The proof follows as for Lemma 2.4 using in addition that $\sigma^j(x)$ is a monotonically nondecreasing function. □

Clearly, the constants $C_P(Y_{\ell_i};X_i)$ in Lemma 3.6 (and thus $C^*_{\mathrm{P},\alpha}$ in Theorem 3.4) are independent of $\{\alpha_k\}_{k=1}^n$. However, to bound them independently of $\mathcal{Y}$ (i.e., geometric parameters), it is necessary to require a certain regularity of the subregions Y_k. This is technical and will be discussed in detail in Section 4.

3.1 *Necessity of the quasi-monotonicity condition*

The following result shows that type-0 quasi-monotonicity, in the sense of Definition 3.2, is in fact a necessary condition for $C_{\mathrm{P},\alpha}(D)$ to remain bounded when the contrast in the coefficient goes to infinity.

PROPOSITION 3.7 Suppose that $\alpha \in L^\infty_+(D)$ satisfies (2.1) and the subregions $\{Y_\ell\}_{\ell=1}^n$ are ordered such that $\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_1$. If α is *not* type-0 quasi-monotone in the sense of Definition 3.2, then there exist indices k, j with $n > k > j \geqslant 1$ and a constant $C > 0$ independent of $\{\alpha_\ell\}_{\ell=1}^n$ such that

$$\alpha_k > \alpha_j \quad \text{and} \quad C_{\mathrm{P},\alpha}(D) \geqslant C\frac{\alpha_k}{\alpha_j},$$

that is, $C_{\mathrm{P},\alpha}(D) \to \infty$ as $\alpha_k/\alpha_j \to \infty$. The same is true for the (best) constant in the discrete weighted Poincaré inequality (3.6).

Proof. Clearly,

$$C_{\mathrm{P},\alpha}(D) \geqslant \sup_{u\in H^1(D)}\frac{\inf_{c\in\mathbb{R}}\|u-c\|^2_{L^2(D),\alpha}}{|u|^2_{H^1(D),\alpha}}. \tag{3.8}$$

If α is not type-0 quasi-monotone (in the sense of Definition 3.2), then there exists an index k, with $n > k > 1$, such that there is no type-0 quasi-monotone path from Y_k to $Y_{\ell^*} = Y_n$. Let us assume that k is the largest index having this property, and let $j \in \{1,\ldots,k-1\}$ be the largest index such that $\alpha_k > \alpha_j$. Then it follows that the regions

$$Y_M := (\bar{Y}_{j+1}\cup\cdots\cup\bar{Y}_k)^\circ \quad \text{and} \quad Y_H := (\bar{Y}_{k+1}\cup\cdots\cup\bar{Y}_n)^\circ$$

are separated by $Y_L := (\bar{Y}_1\cup\cdots\cup\bar{Y}_j)^\circ$ or in other words, $\bar{Y}_M$ and $\bar{Y}_H$ are disconnected. (If they were not, there would be a type-0 quasi-monotone path from Y_k to Y_n).

Now choose $u^* \in H^1(D)$ such that

$$u^*_{|Y_H} = +1, \quad u^*_{|Y_M} = -1 \text{ and } |u^*|^2_{H^1(Y_L)} \leqslant \beta. \tag{3.9}$$

The existence of such a function follows from the fact that the trace operator of a Lipschitz domain has a continuous right inverse (see, for example, McLean, 2000, Theorem 3.37), which yields

$$|u^*|^2_{H^1(Y_L)} \leqslant \|u^*\|^2_{H^1(Y_L)} \leqslant C_{\mathrm{tr}} \|u^*\|^2_{H^1(\partial Y_L \cap (\partial Y_H \cup \partial Y_M))} =: \beta$$

if we choose $u^*_{|Y_L}$ as the right inverse of the trace prescribed on ∂Y_H and ∂Y_M. Since this trace is constant, β depends only on the region Y_L. Note that if either of the regions Y_H and Y_M is not connected (but only type-0 connected; see Remark 3.3), then Y_L may fail to be a Lipschitz domain. However, in this case, the regions Y_H and Y_M (where $u^* = \pm 1$) can be extended suitably such that they are still disconnected and such that the remainder is Lipschitz. This way, we still get property (3.9).

Next, we investigate the numerator and denominator in (3.8) for $u = u^*$. Firstly,

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|u^* - c\|^2_{L^2(D),\alpha} &\geqslant \inf_{c\in\mathbb{R}} \{|1-c|^2 \alpha_{k+1} \operatorname{meas}_d(Y_H) + |1+c|^2 \alpha_k \operatorname{meas}_d(Y_k)\} \\ &\geqslant \inf_{c\in\mathbb{R}} \{|1-c|^2 + |1+c|^2\} \alpha_k \gamma = 2\gamma\alpha_k, \end{aligned} \tag{3.10}$$

where $\gamma := \min(\operatorname{meas}_d(Y_H), \operatorname{meas}_d(Y_k))$. Secondly, to estimate the weighted H^1 norm of u^* from above, note first that the gradient of u^* vanishes on Y_M and on Y_H. And so, using (3.9), we can conclude that

$$|u^*|^2_{H^1(D),\alpha} = |u^*|^2_{H^1(Y_L),\alpha} \leqslant \alpha_j |u^*|^2_{H^1(Y_L)} \leqslant \beta\alpha_j.$$

Together with (3.8) and (3.10), this implies the result with $C = 2\gamma/\beta$.

To get a lower bound for the discrete weighted Poincaré constant in (3.6), we only need to replace $u \in H^1(D)$ in (3.8) by $u \in V^h(D)$. We can bound the supremum from below by choosing the function $\Pi_h u^*$, where u^* is constructed as above and Π_h is a Scott–Zhang interpolant that leaves the value ± 1 in $Y_H \cup Y_M$ unchanged. In this way, one can derive the bound $|\Pi_h u^*|_{H^1(Y_L)} \leqslant \beta$, where β additionally depends on the shape-regularity constant of $\mathcal{T}^h(D)$. The rest of the proof is analogous. □

It is in fact possible to extend this proof to show that only quasi-monotonicity in the sense of Definition 2.1 is *necessary* for $C_{\mathrm{P},\alpha}(D)$ to remain bounded, but the proof is more technical and we omit it.

4. Explicit dependence on geometrical parameters

Before going into the technical details, let us suppose that the partition $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ consists of a few well-shaped subregions and that all the interfaces X_i between adjacent subregions in Definitions 2.1 and 3.2 are well shaped and sufficiently large. Then it follows from classical results that the constants $C_P(Y_{\ell_i}; X_i)$ and $C_P(Y_{\ell_i}; X_{i-1})$ in Definitions 2.3 and 3.5 are benign (in particular, they are independent of $\mathcal{Y}$ and h). Owing to Lemma 2.4 this implies that the constants $C^*_{\mathrm{P},\alpha}$ in the weighted Poincaré inequalities in Theorems 2.2 and 3.4 are also benign.

If the assumptions above do not hold, then

(i) the number n of subregions may be large,

(ii) the shapes of the subregions Y_ℓ may be complicated, in particular long or thin, and/or

(iii) the interfaces may be small compared with adjacent subregions.

In Section 4.1, we allow the number n to become large, but we restrict ourselves to shape-regular simplicial partitions $\mathcal{Y}$ (such that the situations in (ii) and (iii) are ruled out). We can then give explicit bounds for $C^*_{\mathrm{P},\alpha}$ in terms of n and $H/\eta_{\min}$, where

$$\eta_{\min} := \min_{\ell=1}^{n} \operatorname{diam}(Y_\ell),$$

which is a measure of the 'small scale' that the coefficient introduces. In Section 4.2, we generalize the results to type-m quasi-monotone coefficients. In principle this fully describes the dependence of $C^*_{\mathrm{P},\alpha}$ on α, since the situations in (ii) and (iii) can always be overcome by further subdividing some regions until the partition $\mathcal{Y}$ is shape regular. However, this can lead to pessimistic bounds. Therefore, in Sections 4.3–4.5, we show enhanced bounds for a few distinguished cases including anisotropic subregions, subregions with holes, as well as a checkerboard distribution.

For the remainder let us restrict ourselves to the case $d \geqslant 2$ and to piecewise constant weight functions α satisfying (2.1). To simplify the presentation we write $a \lesssim b$ if a/b can be bounded uniformly by a constant C that is independent of any parameters, in particular independent of α, $\mathcal{Y}$, H and h. Furthermore, we write $a \eqsim b$ if $a \lesssim b$ and $b \lesssim a$.

4.1 *Inequalities for shape-regular partitions*

Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a conforming simplicial triangulation of D and define

$$\eta_\ell := \operatorname{diam}(Y_\ell), \quad \eta := \max_{\ell=1}^{n} \eta_\ell, \quad \eta_{\min} := \min_{\ell=1}^{n} \eta_\ell, \tag{4.1}$$

as well as the *shape-regularity constant*

$$c_{\mathrm{reg}}^{\mathcal{Y}} := \max_{\ell=1}^{n} \frac{\operatorname{diam}(Y_\ell)}{\rho(Y_\ell)}. \tag{4.2}$$

Recall that a family $\{\mathcal{Y}_\eta\}_{\eta\in\Xi}$ of simplicial partitions is called *shape regular* if there is a uniform bound for $c_{\mathrm{reg}}^{\mathcal{Y}_\eta}$. It is called *quasi-uniform* if it is shape regular and the ratios $\eta/\eta_{\min}$ are uniformly bounded. With a slight abuse of notation we will call a partition shape regular or quasi-uniform if it is an element of a family of such partitions.

The next lemma bounds the weighted Poincaré constant explicitly in terms of a few geometric parameters. Recall that, for any quasi-monotone $\alpha \in L^\infty_+(D)$ with underlying partitioning $\{Y_\ell\}_{\ell=1}^n$ and $\ell^* = \operatorname{argmax}\{\alpha_\ell\}_{\ell=1}^n$, the length of the quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*} in Definition 2.1 is denoted by s_k.

LEMMA 4.1 Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a shape-regular simplicial partition of D and let $\alpha \in L_+^\infty(D)$ be quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 2.1, with X^* a facet of the simplex Y_{ℓ^*}). Then

$$C_{\mathrm{P},\alpha}^* \leqslant 2^{d+1}(c_{\mathrm{reg}}^{\mathcal{Y}})^{d-1} \sum_{k=1}^{n} \frac{s_k \operatorname{meas}_d(Y_k)}{H^2 \eta_{\min}^{d-2}}.$$

Proof. The proof is based on Lemma 2.4 and we adopt the same notation. We fix $k \in \{1,\dots,n\}$ and choose a quasi-monotone path $P_{k,\ell^*} = (\bar{Y}_{\ell_1} \cup \cdots \cup \bar{Y}_{\ell_{s_k}})^\circ$ of length s_k. It follows from Lemma A1 in Appendix A that $\max\{C_P(Y_{\ell_i}; X_{i-1}), C_P(Y_{\ell_i}; X_i)\} \leqslant 1$. Owing to Lemma A2 in Appendix A,

$$\frac{\operatorname{diam}(Y_\ell)^2}{\operatorname{meas}_d(Y_\ell)} \leqslant 2^{d-1}(c_{\mathrm{reg}}^{\mathcal{Y}})^{d-1}\eta_\ell^{2-d}.$$

Thus, Lemma 2.4 implies that

$$c_k^{X^*} \leqslant 4\sum_{i=1}^{s_k} 2^{d-1}(c_{\mathrm{reg}}^{\mathcal{Y}})^{d-1}\frac{\operatorname{meas}_d(Y_k)}{H^2}\eta_{\ell_i}^{2-d}. \tag{4.3}$$

Since $d \geqslant 2$ the result follows from the definition of $C_{\mathrm{P},\alpha}^*$ in Definition 2.1. □

The following corollary gives the worst-case scenario.

COROLLARY 4.2 With the assumptions of Lemma 4.1,

$$C_{\mathrm{P},\alpha}^* \lesssim (H/\eta_{\min})^{2(d-1)}.$$

If we assume in addition that $s_k \lesssim H/\eta_{\min}\ \forall\, k = 1,\dots,n$ (which is satisfied, for example, when the sum of the diameters of the subregions in each path is $\lesssim H$), then

$$C_{\mathrm{P},\alpha}^* \lesssim (H/\eta_{\min})^{d-1}.$$

Proof. Note that $\sum_{k=1}^n \operatorname{meas}_d(Y_k) = \operatorname{meas}_d(D) \leqslant H^d$. Owing to shape regularity, $s_k \leqslant n \lesssim (H/\eta_{\min})^d$ (at most). Hence, the result follows from Lemma 4.1. □

Obviously, the results above extend straightforwardly to the case of partitions of $\mathcal{Y}$ into polytopes, where each subregion Y_ℓ consists of a small number of simplices, such that the resulting simplicial partition of D is shape-regular and conforming. In the examples below we shall often make use of this fact.

EXAMPLE 4.1 Let $d = 2$ and consider the three domains shown in Fig. 4. Note that in all three cases the assumptions of Lemma 4.1 are fulfilled, the underlying simplicial partition (only shown for (a)) is shape regular, $\operatorname{meas}_2(D) \eqsim H^2$ and $\eta_{\min} \eqsim 2^{-n}H$. Since $\max_{k=1}^n s_k \leqslant n \lesssim \log_2(H/\eta_{\min})$ in each of these cases, it follows from Lemma 4.1 that

$$C_{\mathrm{P},\alpha}^* \lesssim 1 + \log\left(\frac{H}{\eta_{\min}}\right).$$

REMARK 4.3 Example 4.1 shows that the (standard) Poincaré constant $C_{\mathrm{P}}(D)$ of the two-dimensional dumb-bell domain in Fig. 4(c) is $\mathcal{O}(1 + \log(H/\eta_{\min}))$. Note that the isoperimetric constant (often used

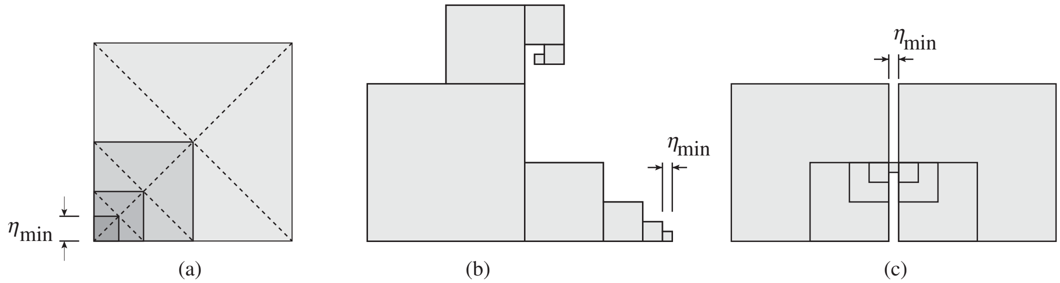


FIG. 4. Some (more complicated) two-dimensional examples with shape-regular partitions. In each case, a corresponding family of partitions is defined by continuing the fractal structure and therefore halving $\eta_{\min}$. In case (a), different colours mean different subregions and the dashed lines indicate how to further subdivide in order to obtain a simplicial partition.

to bound $C_{\mathrm{P}}(D)$; cf. Maz'ja, 1960, 1985; Dohrmann *et al.*, 2008) is $\mathcal{O}(H/\eta_{\min})$ and thus yields a pessimistic bound for $C_{\mathrm{P}}(D)$.

EXAMPLE 4.2 Now let $d=3$ and consider the domain in Fig. 5 (left) with Y_1 being the small cube in the top corner and the remaining subregions numbered away from Y_1, such that $\eta_k \approx 2^k\eta_{\min}$.

Let us first consider the case that $\ell^* = 1$, that is, the largest coefficient is in the small cube. Let k be fixed; then $s_k = k$ and $\ell_i = k+1-i$. It follows from inequality (4.3) that

$$c_k^{X^*} \lesssim \sum_{i=1}^{s_k} \frac{\eta_k^3}{H^2}\eta_{k+1-i}^{-1} \lesssim \frac{4^k\eta_{\min}^2}{H^2}\sum_{i=1}^{k} 2^i \lesssim \frac{\eta_{\min}^2}{H^2}8^k.$$

Since $n \approx \log_2(H/\eta_{\min})$, we get $8^n \approx (H/\eta_{\min})^3$ and thus

$$C_{\mathrm{P},\alpha}^* \lesssim \frac{\eta_{\min}^2}{H^2}\sum_{k=1}^{n} 8^k \lesssim \frac{H}{\eta_{\min}}.$$

For the analogous configuration in higher dimensions, we get $C_{\mathrm{P},\alpha}^* \lesssim (H/\eta_{\min})^{d-2}$.

If, on the other hand, the largest coefficient value is attained in the largest domain, that is, $\ell^* = n$, then, for fixed k, we have $s_k = n-k+1$ and $\ell_i = k-1+i$. And so, again using inequality (4.3), we get (for $d=3$)

$$c_k^{X^*} \lesssim \sum_{i=1}^{n-k+1} \frac{\eta_k^3}{H^2}\eta_{k-1+i}^{-1} \lesssim \frac{\eta_{\min}^2}{H^2}4^k\sum_{i=1}^{n-k+1} 2^{1-i} \lesssim \frac{\eta_{\min}^2}{H^2}4^k \lesssim 4^{k-n},$$

where in the last step we used that $\eta_{\min} \approx 2^{-n}H$. Hence, for any n,

$$C_{\mathrm{P},\alpha}^* \lesssim 1.$$

In the same way, we can also show that $C_{\mathrm{P},\alpha}^* \lesssim 1$ for the domains in Fig. 4(a, b) if the largest coefficient is attained in the largest subregion.

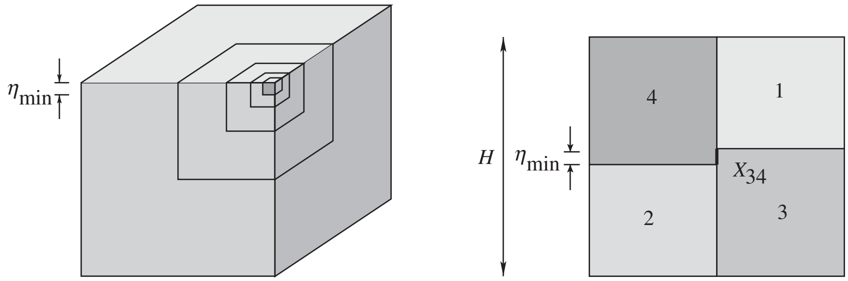


FIG. 5. Left: example with shape-regular polyhedral partition, consisting of a small cube and nested Fichiera corners. Right: coefficient distribution with a staggered structure. (The largest coefficient is in region Y_4.)

Note that the examples in this section are not artificial. They arise naturally when interfaces between perfectly well-shaped coefficient regions are *small* compared with the size of the regions; see, for example, Fig. 5 (right). This case can often be treated by *artificially* subdividing some subregions further in a suitable way.

EXAMPLE 4.3 Consider the scenario in Fig. 5 (right). The quasi-monotone path $P_{3,4}$ from Y_3 to Y_4 contains the interface $X_{3,4}$, which has $\operatorname{diam}(X_{3,4}) = \eta_{\min} \ll H$. However, subdividing both Y_3 and Y_4 further as shown in Fig. 4(a), we get as in Example 4.1

$$C^*_{\mathrm{P},\alpha} \lesssim 1 + \log\left(\frac{H}{\eta_{\min}}\right).$$

4.2 *Inequalities for FE functions on shape-regular partitions*

In this subsection, we generalize the explicit results of the previous section to the discrete case and discuss a few particularities.

It was important in Section 4.1 that the $(d-1)$-dimensional manifold X^* was chosen to be a $(d-1)$-dimensional facet of the simplex Y_{ℓ^*}, that is, an edge in two dimensions or a face in three dimensions. In this section, for type-m quasi-monotone coefficients, we choose X^* to be an *m-facet* of the simplex Y_{ℓ^*}.

DEFINITION 4.4 Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a simplicial partition of D. Then each boundary ∂Y_ℓ is the union of simplicial m-dimensional manifolds called *m-facets*, where $m = 0, \ldots, d-1$. In particular,

(a) 0-*facets*: the vertices of the simplex;

(b) 1-*facets*: the edges of the simplex;

(c) 2-*facets*: the faces of the simplex if $d \geqslant 3$.

It is straightforward to extend the results from Section 4.1 to type-m quasi-monotone coefficients, provided the mesh $\mathcal{T}_h(D)$ resolves the partition $\mathcal{Y}$ and is quasi-uniform on each of the simplices Y_ℓ. Doing this carefully we even get an enhanced bound compared with Theorem 3.4. Let $h_\ell := \max_{\tau\subset Y_\ell} \operatorname{diam}(\tau)$ be the local mesh size on Y_ℓ and recall that s_k is the length of the type-m quasi-monotone path P_{k,ℓ^*} defined in Definition 3.2.

LEMMA 4.5 For $d > 1$, let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a shape-regular simplicial partition of D and let $\mathcal{T}_h(D)$ be such that its restriction $\mathcal{T}_h(Y_\ell)$ is quasi-uniform $\forall\, \ell = 1, \dots, n$. If $\alpha \in L_+^\infty(D)$ is type-m quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and if X^* is an m-facet of the simplex Y_{ℓ^*}, then

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leqslant C^{*,m}_{\mathrm{P},\alpha} H^2 |u|^2_{H^1(D),\alpha} \quad \forall u \in V^h(D), \tag{4.4}$$

where $C^{*,m}_{\mathrm{P},\alpha} \lesssim \sigma^{d-m}(\max_{\ell=1}^n(\eta_\ell/h_\ell)) \sum_{k=1}^n s_k(\mathrm{meas}_d(Y_k)/H^2\eta_{\min}^{d-2})$, where σ^{d-m} is as defined in (3.4). The hidden constant depends on $c_{\mathrm{reg}}^{\mathcal{Y}}$ and on the constant c_{discr} from Lemma 3.1.

Proof. The proof follows the same lines as that of Lemma 4.1. Let $c = \bar{u}^{X^*}$. Since $c_{\mathrm{reg}}^{\mathcal{Y}} \eqsim 1$ it follows from Lemma 3.1 that the constants $C_P(Y_{\ell_i}; X_{i-1})$ and $C_P(Y_{\ell_i}; X_i)$ in the discrete Poincaré inequality (3.7) are bounded by $c_{\mathrm{discr}} = \mathcal{O}(1)$ and the additional factors are $\mathcal{O}(\sigma^{d-m}(\eta_{\ell_i}/h_{\ell_i}))$. From this, one can derive (as in the proof of Lemma 2.4) the local inequality

$$\|u - \bar{u}^{X^*}\|^2_{L^2(Y_k)} \lesssim \left[\sum_{i=1}^{s_k} \frac{\mathrm{meas}_d(Y_k)}{\eta_{\ell_i}^{d-2}} \sigma^{d-m}\left(\frac{\eta_{\ell_i}}{h_{\ell_i}}\right)\right] |u|^2_{H^1(P_{k,\ell^*})} \quad \forall u \in V^h(P_{k,\ell^*}), \tag{4.5}$$

which is an analogue to (4.3) (see also (2.2)). The result then follows as in the proof of Lemma 4.1. □

As in Section 4.1, if we exclude pathological examples with type-m quasi-monotone paths P_{k,ℓ^*} that follow convoluted curves with length $\gg H$, Lemma 4.5 yields a worst-case scenario of

$$C^{*,m}_{\mathrm{P},\alpha} \lesssim \left(\frac{H}{\eta_{\min}}\right)^{d-1} \sigma^{d-m}\left(\max_{\ell=1}^n \frac{\eta_\ell}{h_\ell}\right).$$

To apply Lemma 4.1 it was crucial that each Y_ℓ in the partition was a simplex. As mentioned several times, a polytope Y_ℓ that is not simplicial can always be artificially subdivided into a set of simplicial ones. However, it is often difficult to guarantee that a mesh $\mathcal{T}_h(D)$ that is aligned with the original partition is also aligned with the artificial simplicial subpartition, and we would not want to impose such a condition. The next lemma shows that, for any polytope Y that is the union of a small number of simplices, it suffices that there exists a quasi-uniform triangulation $\tilde{\mathcal{T}}_h(Y)$ that is aligned with the simplicial subpartition of Y and has the same mesh size as $\mathcal{T}_h(Y)$ such that the results of Lemma 4.1 hold.

LEMMA 4.6 Let Y be the union of a small number of shape-regular and quasi-uniform simplices $T_1, \dots, T_p$ with diameter $\mathrm{diam}(T_i) \eqsim H := \mathrm{diam}(Y)$. Let $\mathcal{T}_h(Y)$ be a quasi-uniform simplicial triangulation of Y (not necessarily aligned with $\{T_i\}_{i=1}^p$) and let $X \subset \partial Y$ be an m-facet of one of the simplices T_i (note that X is resolved by $\mathcal{T}_h(Y)$). Then

$$\|u - \bar{u}^X\|^2_{L^2(Y)} \lesssim \sigma^{d-m}\left(\frac{H}{h}\right) H^2 |u|^2_{H^1(Y)} \quad \forall u \in V_h(Y).$$

The hidden constant depends on the number of simplices p, on the constant c_{discr} in Lemma 3.1 and on the shape-regularity constants of $\mathcal{T}_h(Y)$ and $\{T_i\}_{i=1}^p$.

Proof. It is always possible to refine the simplices $T_1, \dots, T_p$ to obtain a quasi-uniform simplicial triangulation $\tilde{\mathcal{T}}_h(Y)$ with mesh size h that coincides with $\mathcal{T}_h(Y)$ on the boundary ∂Y and that has a shape-regularity constant that is bounded by the shape-regularity constants of $\mathcal{T}_h(Y)$ and $\{T_i\}_{i=1}^p$. Let $\tilde{V}_h(Y)$

be the corresponding FE space of continuous, piecewise linear functions. Since $\tilde{\mathcal{T}}_h(Y)$ is aligned with $\{T_i\}_{i=1}^p$ we can apply Lemma 4.5 (with $\alpha \equiv 1$) to get

$$\|u - \bar{u}^X\|_{L^2(Y)}^2 \lesssim \sigma^{d-m}\left(\frac{H}{h}\right) H^2 |u|_{H^1(Y)}^2 \quad \forall u \in \tilde{V}_h(Y). \tag{4.6}$$

To show that an equivalent statement holds for functions $u \in V_h(Y)$ we make use of the Scott–Zhang operator from Scott & Zhang (1990) (see also Brenner & Scott, 2002). Let $V_h(\partial Y)$ be the trace space of $V_h(Y)$, which is identical to the trace space of $\tilde{V}_h(Y)$. There exists an operator $\Pi_h : H^1(Y) \to \tilde{V}_h(Y)$ such that, $\forall\, v \in H^1(Y)$ with $v_{|\partial Y} \in V_h(\partial Y)$,

$$(\Pi_h v)_{|\partial Y} = v_{|\partial Y}, \tag{4.7}$$

$$\|v - \Pi_h v\|_{L^2(Y)} \leqslant C_{\mathrm{sc}} h |v|_{H^1(Y)}, \tag{4.8}$$

$$|\Pi_h v|_{H^1(Y)} \leqslant C_{\mathrm{sc}} |v|_{H^1(Y)}. \tag{4.9}$$

The operator is constructed by local averages over $(d-1)$-dimensional manifolds and the constant C_{sc} only depends on the shape-regularity constant of $\tilde{\mathcal{T}}_h(Y)$.

Let $u \in V_h(Y)$ be arbitrary but fixed. Then, due to (4.7), $\overline{\Pi_h u}^X = \bar{u}^X$ and it follows from (4.6) and (4.8) that

$$\begin{aligned} \|u - \bar{u}^X\|_{L^2(Y)} &\leqslant \|u - \Pi_h u\|_{L^2(Y)} + \|\Pi_h u - \overline{\Pi_h u}^X\|_{L^2(Y)} \\ &\lesssim h|u|_{H^1(Y)} + \sqrt{\sigma^{d-m}\left(\frac{H}{h}\right)} H |\Pi_h u|_{H^1(Y)}. \end{aligned}$$

Clearly, $h \leqslant H$ and $\sigma^{d-m}(H/h) \geqslant 1$, and so the result follows from (4.9). □

4.3 *Anisotropic subregions*

In this subsection, we treat cases where the partition $\mathcal{Y}$ contains anisotropic subregions. We will see that it is often advantageous *not* to further subdivide this into a shape-regular partition. We start by showing an elementary result for the Poincaré constant of a parallelepiped; see also Veeser & Verfürth (2012, Section 2.1).

LEMMA 4.7 Let $\{\vec{e}_i\}_{i=1}^d$ be a (normalized) basis of $\mathbb{R}^d$ and let Y be the parallelogram/parallelepiped $\{\sum_{i=1}^d \beta_i \vec{e}_i : \beta_i \in (0, L_i)\}$. If X is one of the facets (edges/faces) of Y, then

$$C_P(Y; X) \eqsim 1,$$

and the hidden constant is independent of the aspect ratios L_i/L_j and of the angles between $\vec{e}_i$ and $\vec{e}_j$ for any $1 \leqslant i, j \leqslant d$.

Proof. The result can easily be shown by transforming Y to the (isotropic) reference cube $Q = (0,1)^d$ using the *linear* transformation $F(x) = J^{-1}x$, where $J = (L_1\vec{e}_1|\cdots|L_d\vec{e}_d)$. □

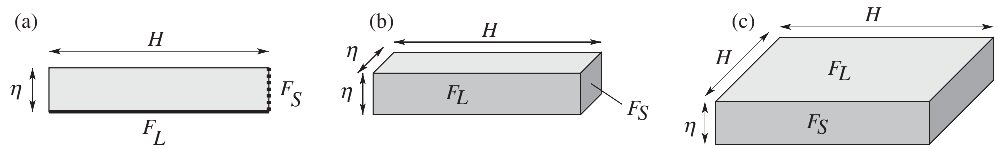


FIG. 6. Three model cases of anisotropic domains in two (case (a)) and three dimensions (cases (b) and (c)).

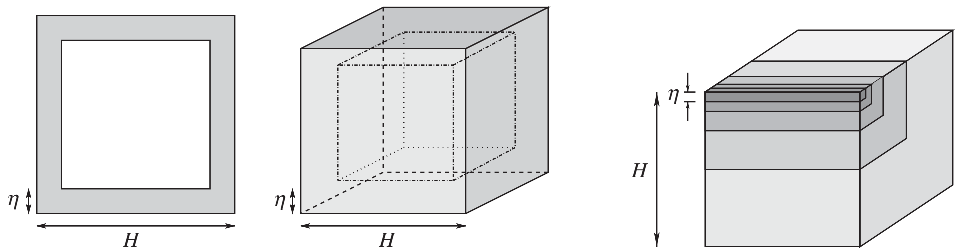


FIG. 7. Left/middle: 'annular' subregions in Example 4.5 in two and three dimensions. The smaller cube sketched inside is cut out from the larger cube. Right: piecewise constant coefficient distribution increasing gradually towards an edge in three dimensions.

EXAMPLE 4.4 For any of the regions Y in Fig. 6(a,b,c) and for any $(d-1)$-facet X of Y, Lemma 4.7 implies

$$C(Y,X) \approx 1,$$

independent of the aspect ratio H/η.

EXAMPLE 4.5 Let Y be one of the two 'annular' subregions shown in Fig. 7 (left, middle), and let X be an edge of length H (left figure) or a face of area H^2 (middle figure). Then $C_P(Y;X) \approx 1$. This can be shown by further subdividing the subregions into a few anisotropic rectangles/cuboids and using Lemma 2.4 (with $D = Y$ and $X^* = X$) together with the estimates in Example 4.4. Such estimates can already be found in Pechstein & Scheichl (2008).

Our next example will be Fig. 7 (right), where a piecewise constant coefficient increases gradually towards an edge of a cube in three dimensions. To get an optimal bound in this case is surprisingly difficult. We require a variation of Lemma 2.4.

LEMMA 4.8 Let $\alpha \in L^\infty_+(D)$ be quasi-monotone with respect to a partition $\mathcal{Y}$. Let ℓ^* be the index of the region where the maximum is attained and let X^* be a $(d-1)$-dimensional manifold in ∂Y_{ℓ^*}. For each $k = 1, \ldots, n$, let X_k be a $(d-1)$-dimensional manifold in ∂Y_k and let P_{k,ℓ^*} be the quasi-monotone path from Definition 2.1. Then

$$C^*_{\mathrm{P},\alpha} \lesssim \max_{k=1}^{n} \left\{ \frac{\mathrm{diam}(Y_k)^2}{H^2} C_P(Y_k;X_k) \right\} + \sum_{k=1}^{n} \frac{\mathrm{meas}_d(Y_k)}{\mathrm{meas}_d(P_{k,\ell^*})} \frac{\mathrm{diam}(P_{k,\ell^*})^2}{H^2} \{C_P(P_{k,\ell^*};X_k) + C_P(P_{k,\ell^*};X^*)\}.$$

Proof. Let $1 \leqslant k \leqslant n$ be fixed. Then

$$\tfrac{1}{2}\|u - \bar{u}^{X^*}\|^2_{L^2(Y_k),\alpha} \leqslant \alpha_k \|u - \bar{u}^{X_k}\|^2_{L^2(Y_k)} + \alpha_k\, \mathrm{meas}_d(Y_k)|\bar{u}^{X_k} - \bar{u}^{X^*}|^2. \tag{4.10}$$

For the first term on the right-hand side of (4.10) we have

$$\alpha_k \|u - \bar{u}^{X_k}\|^2_{L^2(Y_k)} \leqslant C_P(Y_k; X_k) \frac{\operatorname{diam}(Y_k)^2}{H^2} H^2 |u|^2_{H^1(Y_k),\alpha}.$$

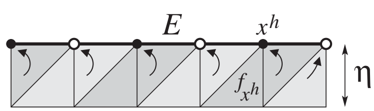
The second term can be bounded in the same way as in (2.7), but using two Poincaré inequalities on the whole of P_{k,ℓ^*} (instead of Y_{ℓ_i}) with manifolds X_k and X^* (instead of X_{i-1} and X_i, respectively). To conclude the proof we use quasi-monotonicity and sum the two bounds over $k = 1, \ldots, n$. □

EXAMPLE 4.6 For the scenario in Fig. 7 (right), we have

$$C^*_{\mathrm{P},\alpha} \lesssim \left(1 + \log\left(\frac{H}{\eta}\right)\right)^2.$$

To see this, we first consider the subdivision $\{Y_\ell\}_{\ell=1}^n$ with $n \eqsim 1 + \log(H/\eta)$ depicted in Fig. 7 (right) and apply Lemma 4.8 with X^* one of the long and thin faces of Y_{ℓ^*}. Clearly, $C_P(Y_k; X_k) \eqsim 1$ and $C_P(P_{k,\ell^*}; X_k) \eqsim 1$ because these regions consist of a few cuboids and X_k is one of the faces. Hence, it remains to investigate $C_P(P_{k,\ell^*}; X^*)$. First we consider the case $k = 1$, where $P_{1,\ell^*} = D$.

In the limit case $\eta \to 0$, the face X^* collapses to an edge E of D. Here we can make use of Lemma 3.1 which can straightforwardly be generalized to cubes. Let $\mathcal{T}_h$ be an auxiliary *uniform Cartesian* triangulation of D such that the face X^* is resolved by just one layer of element faces ($h \eqsim \eta$); see also X^* in figure 8.



Let $V_h(D)$ denote the corresponding piecewise linear FE space. As in Lemma 4.6, we make use of a Scott–Zhang-type quasi-interpolation operator (see Scott & Zhang, 1990; Brenner & Scott, 2002), that is, there exists an operator $\Pi_h : H^1(D) \to V_h(D)$ such that, $\forall v \in H^1(D)$,

$$\|v - \Pi_h v\|_{L^2(D)} \leqslant C_{\mathrm{sc}} h |v|_{H^1(D)},$$
$$|\Pi_h v|_{H^1(D)} \leqslant C_{\mathrm{sc}} |v|_{H^1(D)},$$

with a uniform constant C_{sc}. The interpolator is constructed by defining the value at each mesh node by the average over a suitable $(d-1)$-dimensional manifold, typically a face associated to the node. The choice of these manifolds is quite arbitrary. However, for each node x^h in $\bar{E}$, we choose the associated manifold f_{x^h} as displayed in the figure above. Note that $\sum_{x^h \in \bar{E}} |f_{x^h}| = |X^*|$, and thus,

$$\int_E \Pi_h v \, \mathrm{d}s = \sum_{x^h \in \bar{E}} \frac{|f_{x^h}|}{\eta} \bar{v}^{f_{x^h}} = \frac{1}{\eta} \sum_{x^h \in \bar{E}} \int_{f_{x^h}} v \, \mathrm{d}s = \frac{1}{\eta} \int_{X^*} v \, \mathrm{d}s \quad \forall v \in H^1(D).$$

Since $|X^*| = \eta |E|$, it follows that

$$\overline{\Pi_h v}^E = \bar{v}^{X^*} \quad \forall v \in H^1(D).$$

We now obtain from the properties of the operator Π_h constructed above and from Lemma 3.1 that, $\forall\, u \in H^1(D)$,

$$\begin{aligned}\|u-\bar{u}^{X^*}\|^2_{L^2(D)} &\lesssim \|u-\Pi_h u\|^2_{L^2(D)} + \|\Pi_h u - \overline{\Pi_h u}^E\|^2_{L^2(D)} \\ &\lesssim h^2|u|^2_{H^1(D)} + H^2(1+\log(H/h))|\Pi_h u|^2_{H^1(D)} \\ &\lesssim H^2(1+\log(H/h))|u|^2_{H^1(D)}.\end{aligned}$$

Hence, since $h \eqsim \eta$, we get that $C_P(P_{1,\ell^*};X^*) \lesssim 1+\log(H/\eta) \eqsim n$.

Next we investigate P_{k,ℓ^*} for $k>1$. Consider the linear transformation from the reference cube $\hat{Q}$ to P_{k,ℓ^*}. This consists simply in multiplying two of the coordinates by $2^{-k}H^{-1}$ and the remaining one by H^{-1}. Then

$$\|\hat{u}\|^2_{L^2(\hat{Q})} = \frac{\mathrm{meas}_d(\hat{Q})}{\mathrm{meas}_d(P_{k,\ell^*})}\|u\|^2_{L^2(P_{k,\ell^*})} \quad \text{and} \quad |\hat{u}|^2_{H^1(\hat{Q})} \leqslant \frac{\mathrm{meas}_d(\hat{Q})}{\mathrm{meas}_d(P_{k,\ell^*})}|u|^2_{H^1(P_{k,\ell^*})},$$

because the spectral norm of the Jacobian is $\leqslant 1$. On $\hat{Q}$ we can choose a quasi-uniform mesh with mesh size $h \eqsim 2^{-(n+1-k)}$ and apply the arguments from the case $k=1$ (with $D=\hat{Q}$ and $H=1$) to obtain

$$C_P(P_{k,\ell^*};X^*) \lesssim 1+\log(1/h) \eqsim 1+\log(2^{n+1-k}) \eqsim n+1-k.$$

Putting all the estimates together and using Lemma 4.8 finally yields

$$C^*_{\mathrm{P},\alpha} \lesssim 1 + \sum_{k=1}^{n}\frac{4^{-k}H^3H^2}{4^{-k}H^3H^2}(1+(n+1-k)) \eqsim \sum_{k=1}^{n}(2+n-k) \eqsim n^2,$$

where $n \eqsim 1+\log(H/\eta)$. Thus, we have shown that $C^*_{\mathrm{P},\alpha} \lesssim (1+\log(H/\eta))^2$. Note that the numerical results in Section 5.4 show that this result is not quite sharp. They suggest a behaviour of $\mathcal{O}(1+\log(H/\eta))$.

Unfortunately, using Lemma 4.8 for the layered coefficient distribution in Fig. 8 (left, middle) leads to a suboptimal bound $C^*_{\mathrm{P},\alpha} \lesssim 1+\log(H/\eta)$ (that grows with the number of layers). The following alternative theory to Lemmas 2.4 and 4.8 (first given in Pechstein & Scheichl, 2008, Appendix) leads to optimal bounds even in these cases.

Here, we actually do need to further partition the anisotropic subregions such that $\{Y_\ell\}_{\ell=1}^n$ is simplicial and quasi-uniform. Furthermore, X^* has to be the union of a subset $\{F_j\}_{j=1}^J$ of the $(d-1)$-facets of the simplices Y_ℓ (edges for $d=2$ and faces for $d=3$). For simplicity we assume that the numbering is such that Y_j is the (unique) simplex whose boundary contains F_j $\forall\, j=1,\dots,J$.

LEMMA 4.9 Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be simplicial and quasi-uniform with mesh size $\eta>0$, and let $\bar{X}^* = \bigcup_{j=1}^J \bar{F}_j$ such that $F_j \subset \bar{Y}_j$. For any $k \in \mathcal{I} := \{1,\dots,n\}$ and $j \in \mathcal{J} := \{1,\dots,J\}$, let $P_{k,j}$ be a path from Y_k to Y_j. Then

$$\int_{\mathrm{F}_j}\int_{Y_k}|u(x)-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x \lesssim s_{k,j}\eta^{d+1}|u|^2_{H^1(P_{k,j})} \quad \forall u \in H^1(P_{k,j}),$$

where $s_{k,j}$ is the length of the path $P_{k,j}$.

Proof. Note first that

$$\int_{\mathrm{F}_j}\int_{Y_k}|u(x)-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x \lesssim \int_{\mathrm{F}_j}\int_{Y_k}|u(x)-\bar{u}^{F_j}|^2+|\bar{u}^{F_j}-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x$$
$$\lesssim \mathrm{meas}_{d-1}(F_j)\|u-\bar{u}^{F_j}\|^2_{L^2(Y_k)}+\mathrm{meas}_d(Y_k)\|u-\bar{u}^{F_j}\|^2_{L^2(F_j)}. \tag{4.11}$$

It follows from (4.3) together with the definition of $c_k^{X^*}$ in (2.2) (with $X^*=F_j$) that

$$\|u-\bar{u}^{F_j}\|^2_{L^2(Y_k)} \lesssim s_{k,j}\frac{\mathrm{meas}_d(Y_k)}{\eta^{d-2}}|u|^2_{H^1(P_{k,j})}. \tag{4.12}$$

Also, by transformation to the reference simplex we get that

$$\|u-\bar{u}^{F_j}\|^2_{L^2(F_j)} \lesssim \eta|u|^2_{H^1(Y_j)}. \tag{4.13}$$

Substituting these last two bounds into (4.11), the final result follows from the fact that by assumption, $\mathrm{meas}_d(Y_k)\eqsim\eta^d$ and $\mathrm{meas}_{d-1}(F_j)\eqsim\eta^{d-1}$. □

LEMMA 4.10 Under the assumptions of Lemma 4.9, let $\alpha\in L^\infty_+(D)$ be quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and let each $P_{k,j}$ be quasi-monotone with respect to α. Then

$$C^*_{\mathrm{P},\alpha}\lesssim\frac{s_{\max}r_{\max}\eta^{d+1}}{\mathrm{meas}_{d-1}(X^*)H^2},$$

where $s_{\max}:=\max\{s_{k,j}:(k,j)\in\mathcal{I}\times\mathcal{J}\}$ and

$$r_{\max}:=\max_{i\in\mathcal{I}}|\{(k,j)\in\mathcal{I}\times\mathcal{J}:Y_i\subset P_{k,j}\}|,$$

that is, the maximum number of times any of the simplices Y_i is contained in a path.

Proof. Without loss of generality, let $u\in H^1(D)$ with $\bar{u}^{X^*}=0$ be arbitrary but fixed. We now integrate the identity $u(x)^2-2u(x)u(y)+u(y)^2=(u(x)-u(y))^2$ over X^* with respect to x, multiply it by $\alpha(y)$, and finally integrate over D with respect to y:

$$\int_D\alpha(y)\,\mathrm{d}y\|u\|^2_{L^2(X^*)}-2\int_{X^*}u(x)\,\mathrm{d}s_x\int_D\alpha(y)u(y)\,\mathrm{d}y+\mathrm{meas}_{d-1}(X^*)\|u\|^2_{L^2(D),\alpha}$$
$$=\int_{X^*}\int_D\alpha(y)|u(x)-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x.$$

The middle term on the left-hand side vanishes since $\bar{u}^{X^*}=0$. Thus,

$$\mathrm{meas}_{d-1}(X^*)\|u\|^2_{L^2(D),\alpha}\leqslant\int_{X^*}\int_D\alpha(y)|u(x)-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x$$
$$=\sum_{k\in\mathcal{I}}\sum_{j\in\mathcal{J}}\alpha_k\int_{\mathrm{F}_j}\int_{Y_k}|u(x)-u(y)|^2\,\mathrm{d}y\,\mathrm{d}s_x.$$

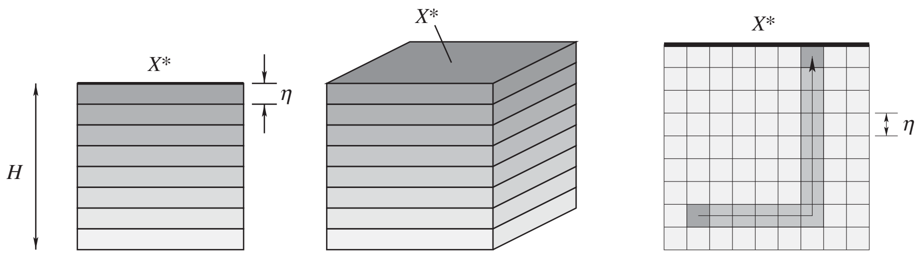


FIG. 8. Left/middle: layered coefficient distributions in two and three dimensions. Right: partitioning and quasi-monotone paths for Example 4.7.

Using Lemma 4.9, quasi-monotonicity and the definitions of $s_{\max}$ and $r_{\max}$,

$$\begin{aligned}\mathrm{meas}_{d-1}(X^*)\|u\|^2_{L^2(D),\alpha} &\lesssim \sum_{k\in\mathcal{I}}\sum_{j\in\mathcal{J}} s_{k,j}\eta^{d+1}|u|^2_{H^1(P_{k,j}),\alpha}\\ &\leqslant s_{\max}\eta^{d+1}\sum_{i\in\mathcal{I}}|\{(k,j)\in\mathcal{I}\times\mathcal{J} : Y_i\subset P_{k,j}\}||u|^2_{H^1(Y_i),\alpha}\\ &\leqslant s_{\max}r_{\max}\eta^{d+1}|u|^2_{H^1(D),\alpha},\end{aligned}$$

which concludes the proof. □

Obviously, the statements of Lemmas 4.9 and 4.10 apply also to nonsimplicial partitions (for example, quadrilateral or hexahedral) if each region Y_i, $i\in\mathcal{I}$ consists of a few simplices and the resulting simplicial mesh is quasi-uniform.

EXAMPLE 4.7 For the two scenarios in Fig. 8 (left, middle), we have

$$C^*_{\mathrm{P},\alpha}\lesssim 1.$$

We only give the proof for $d=2$. The case $d=3$ is analogous.

We subdivide each anisotropic region in Fig. 8 (left) such that the resulting partition $\mathcal{Y}$ consists of $(H/\eta)^2$ square regions Y_k, as shown in Fig. 8 (right). The manifold X^* (on the top of ∂D) with $\mathrm{meas}_{d-1}(X^*)=H$ is the union of H/η edges F_j. By using generic L-shaped paths $P_{k,j}$ from Y_k to F_j as depicted in Fig. 8 (right), for any pair $(k,j)\in\mathcal{I}\times\mathcal{J}$, it is easy to see that (a) each of the paths is quasi-monotone with respect to the given coefficient distribution in Fig. 8 (left), (b) $s_{\max}\approx H/\eta$ and (c) $r_{\max}\approx(H/\eta)^2$. Therefore, it follows from Lemma 4.10 that $C^*_{\mathrm{P},\alpha}\lesssim 1$.

4.4 *Subregions with inclusions*

As an example of this type we consider the region depicted in Fig. 1(c) with a large number of square inclusions with coefficient α_1, inside a background medium with coefficient $\alpha_2>\alpha_1$. We choose X^* to be a boundary edge of D of length $\approx H$.

To bound the weighted Poincaré constant $C^*_{\mathrm{P},\alpha}$ for this case, we treat all the inclusions as one subregion Y_1 and the remainder as Y_2. We choose the path $P_{12}=D$ and bound directly the constants $c_1^{X^*}$, $c_2^{X^*}$ in (2.2). This is still well defined in the case of the disconnected regions Y_1 because the path P_{12} is

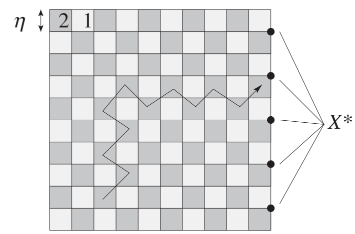


FIG. 9. The checkerboard distribution.

connected. It is easy to see that

$$c_1^{X^*} \lesssim C_P(D, X^*) \lesssim 1.$$

To handle the *perforated* domain Y_2 without the inclusions, we shall use Lemma 4.10 to bound $c_2^{X^*} = C_P(Y_2; X^*)$. It is straightforward to find a quasi-uniform (square) partition $\{\tilde{Y}_i\}_{i=1}^n$ of Y_2 with mesh size equal to the diameter η of the holes (Fig. 1(c)). We construct a (quasi-monotone) path from each region $\tilde{Y}_i$ to one of the faces $F_j \subset X^*$ by following (essentially) the same construction as in Example 4.7 (with some small modifications at the start and at the end of the path). It is easy to see that again $s_{\max} \lesssim H/\eta$ and $r_{\max} \lesssim (H/\eta)^2$. Hence,

$$c_2^{X^*} = C_P(Y_2; X^*) \lesssim 1 \quad \text{and so } C^*_{\mathrm{P},\alpha} \lesssim 1.$$

If there are p distinct values in the inclusions, which are all smaller than α_2, following the same technique we see that

$$C^*_{\mathrm{P},\alpha} \lesssim p.$$

On first glance this would suggest that, in the worst case, $C^*_{\mathrm{P},\alpha} \lesssim n$, but this is not quite true. Using the concept of macroscopically quasi-monotone coefficients (introduced in Section 2.2), we may combine subregions with weights of similar size, even if they are not connected. Assume, for example, that the values of α range from $\alpha_1 = 10^{-6}$ to $\alpha_n = 1$, where Y_n is now the perforated (background) region. If we combine all subregions with values in $[10^{-i}, 10^{-i+1}]$ into one subregion, we have a local variation of 10 in each subregion. Therefore, since there are six such combined subregions,

$$C^*_{\mathrm{P},\alpha} \lesssim 60$$

uniformly, even for $n \to \infty$. We note that estimates for $C^*_{\mathrm{P},\alpha}$ for this example have been shown in Galvis & Efendiev (2010a, Lemma 4), but they depend on the number n of inclusions and are not explicit in the geometric parameters.

4.5 *The checkerboard distribution*

Our last type of example is that of checkerboard-type distributions, as depicted in Fig. 9. We will show that the discrete Poincaré inequality (4.4) for the coefficient in Fig. 9 holds with

$$C^{*,m}_{\mathrm{P},\alpha} \lesssim 1 + \log\left(\frac{\eta}{h}\right).$$

In a similar way to Lemma 4.10 we can prove the following bound for $C_{\mathrm{P},\alpha}^{*,m}$ in (4.4) in Lemma 4.5.

LEMMA 4.11 For $d > 1$, let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a quasi-uniform simplicial partition of D with mesh size $\eta > 0$, and let $\mathcal{T}_h(D)$ be a quasi-uniform refinement of $\mathcal{Y}$ with mesh size $\eta \geqslant h > 0$. If $\alpha \in L_+^\infty(D)$ is type-m quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and X^* is a finite union of type-m facets F_j of the partition $\mathcal{Y}$ (not necessarily connected) such that $\bar{X}^* = \bigcup_{j\in\mathcal{J}} F_j$, then

$$C_{\mathrm{P},\alpha}^{*,m} \lesssim \sigma^{d-m}\left(\frac{\eta}{h}\right)\frac{s_{\max} r_{\max}\eta^{m+2}}{\mathrm{meas}_m(X^*)\,\mathrm{diam}(Y)^2},$$

where $s_{\max}$ and $r_{\max}$ are defined as in Lemma 4.10 for type $(d-1)$.

Proof. Assume first that $m > 0$. By following the proof of Lemma 4.9, we can show that, for any type-m quasi-monotone $P_{k,j}$ from Y_k to Y_j, such that $F_j \subset \bar{Y}_j$, we have

$$\int_{\mathrm{F}_j}\int_{Y_k} |u(x) - u(y)|^2 \,\mathrm{d}y\,\mathrm{d}s_x \lesssim s_{k,j}\eta^{m+2}\sigma^{d-m}\left(\frac{\eta}{h}\right)|u|^2_{H^1(P_{k,j})} \quad \forall u \in V^h(P_{k,j}). \tag{4.14}$$

This result is obtained by using (4.5) instead of (4.3) and the inequality

$$\|u - \bar{u}^{F_j}\|^2_{L^2(F_j)} \lesssim \eta^{2-d+m}\sigma^{d-m}\left(\frac{\eta}{h}\right)|u|^2_{H^1(Y_j)} \quad \forall u \in V^h(Y_j)$$

instead of (4.13). Using (4.14), we simply follow the proof of Lemma 4.10 and obtain the desired bound for $C_{\mathrm{P},\alpha}^{*,m}$.

Finally, we treat the case $m = 0$, which means that X^* is the union of isolated points $F_j = p_j, j \in \mathcal{J}$. Recall the notation $\bar{u}^{X^*} = (1/\mathrm{meas}_0(X^*))\sum_{j\in\mathcal{J}} u(p_j)$ introduced in Section 2.1 for the case $m = 0$, where $\mathrm{meas}_0(X^*) = \sum_{j\in\mathcal{J}} 1$. Similarly, we define $\int_{X^*} v\,\mathrm{d}s := \sum_{j\in\mathcal{J}} v(p_j)$. With this notation, all the arguments from above apply to the case $m = 0$ as well. □

EXAMPLE 4.8 In the two-dimensional checkerboard example in Fig. 9, we assume that the coefficient takes two values, α_1 and $\alpha_2 \gg \alpha_1$. We choose X^* as the union of $\mathcal{O}(H/\eta)$ vertices on the boundary of D, as shown, and construct type-0 quasi-monotone paths $P_{k,j}$ from every square $Y_k \in \mathcal{Y}$ to every vertex $F_j \in X^*$, as shown in the figure. As in Example 4.7 and in Section 4.4, it is easy to see that these paths satisfy $s_{\max} \lesssim H/\eta$ and $r_{\max} \lesssim (H/\eta)^2$, and so, since $\mathrm{meas}_0(X^*) \approx H/\eta$, we finally get from Lemma 4.11 that

$$C_{\mathrm{P},\alpha}^{*,m} \lesssim \sigma^2\left(\frac{\eta}{h}\right)\frac{H}{\eta}\frac{H^2}{\eta^2}\frac{\eta^2}{H/\eta H^2} = 1 + \log\left(\frac{\eta}{h}\right).$$

5. Numerical results

In this section, we compute for some examples, approximations of the weighted Poincaré constant $C_{\mathrm{P},\alpha}(D)$ by computing the smallest nonzero eigenvalue of the generalized eigenvalue problem,

$$K_h\underline{u}_h = \lambda M_h\underline{u}_h.$$

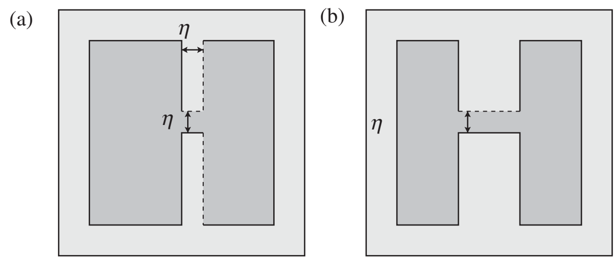


FIG. 10. Two classes of dumb-bell coefficient distributions. Dashed lines indicate variable interfaces for changing η.

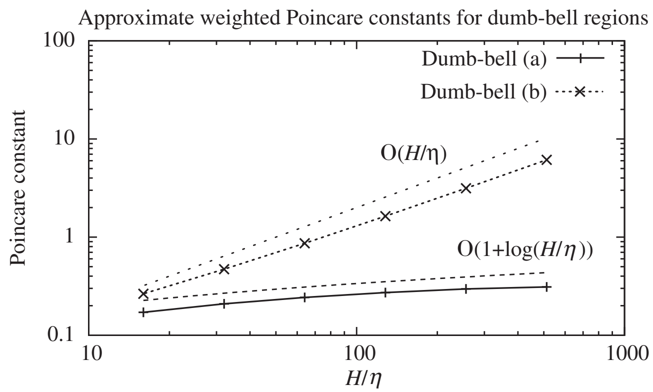


FIG. 11. Approximate Poincaré constants for the dumb-bell distributions in Fig. 10(a, b) for different parameters η.

Here K_h is the α-weighted stiffness matrix, M_h is the α-weighted mass matrix and $\underline{u}_h$ is the coefficient vector of the continuous, piecewise linear FE approximation $u_h \in V_h(D)$ to the corresponding eigenfunction in (1.6)–(1.7) on a suitable mesh $\mathcal{T}_h(D)$. For the eigencomputations we have used the LOBPCG algorithm (Knyazev, 2001) with a factorization of $(K_h + M_h)^{-1}$ as a preconditioner. For the factorization we have used PARDISO (Schenk & Gärtner, 2004, 2006).

5.1 *Dumb-bell-type coefficients*

Here we study the two dumb-bell-type coefficient distributions on $D = (0,1)^2$ shown in Fig. 10. In each particular case, a suitable shape-regular partition $\{Y_\ell\}_{\ell=1}^n$ can be found such that the following holds.

Case (a). As in Fig. 4(c) and Example 4.1, $s_{\max} \approx 1 + \log(H/\eta)$, and so Lemma 4.1 implies $C_P^* \lesssim 1 + \log(H/\eta)$.

Case (b). $s_{\max} \approx H/\eta$, and so $C_P^* \lesssim H/\eta$.

Figure 11 shows the approximate Poincaré constants for $\alpha = 10^5$ inside the dumb-bell and $\alpha = 1$ otherwise. We used a uniform simplicial grid $\mathcal{T}_h(D)$ with $2 \times 512 \times 512$ elements. As we can see, our bounds are sharp and for the considered range of $\eta \in [\frac{1}{16}, \frac{1}{256}]$, the Poincaré constants are always bounded by 10 (even for Case (b)).

TABLE 1 *Discrete weighted Poincaré constants for the checkerboard distribution for various choices of η and h*

$\eta_{\min}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
$h=\frac{1}{4}$	0.07344	—	—	—	—	—	—	—
$\frac{1}{8}$	0.1083	0.05777	—	—	—	—	—	—
$\frac{1}{16}$	0.1466	0.0799	0.05339	—	—	—	—	—
$\frac{1}{32}$	0.1852	0.1061	0.07223	0.05189	—	—	—	—
$\frac{1}{64}$	0.2240	0.1331	0.09518	0.06961	0.05125	—	—	—
$\frac{1}{128}$	0.2629	0.1604	0.1191	0.09146	0.06852	0.05095	—	—
$\frac{1}{256}$	0.3017	0.1876	0.1432	0.1143	0.08991	0.06802	0.05080	—
$\frac{1}{512}$	0.3406	0.2150	0.1674	0.1374	0.1123	0.08921	0.06778	0.05073

5.2 *Checkerboard distribution*

In Section 4.5, we showed that in the case of the checkerboard distribution in Fig. 9, the discrete weighted Poincaré constant in (4.4) can be bounded independently of α by

$$C_{\mathrm{P},\alpha}^{*,m} \lesssim 1 + \log\left(\frac{\eta}{h}\right).$$

We can observe this behaviour in Table 1 for the case $\alpha_1 = 1$ and $\alpha_2 = 10^5$. Keeping η fixed and decreasing h by a constant factor $\frac{1}{2}$ each time, we see a constant additive growth in the Poincaré constant. Also, when η/h is constant, which corresponds to diagonals in the table, the Poincaré constant does not change significantly.

5.3 *Layers*

To study the scenario in Fig. 8 (middle), we choose $\Omega = (0,1)^3$. For n layers (of equal width) we set α to $10^{5((i-1)/(n-1))}$ in the ith layer, where $i = 1,\dots,n$. On a mesh with $32\times32\times32$ elements and varying n from 2 to 32, the computed weighted Poincaré constant is always 0.0337466, which illustrates that it is completely independent of the number of layers.

5.4 *Coefficients growing towards an edge*

Here we study Example 4.6; see also Fig. 7 (right). We choose $\Omega = (0,1)^3$ and let α grow towards the edge of the cube. Let η denote the smallest width of the region of the largest coefficient, as in Fig. 7 (right). Figure 12 (left) shows the coefficient distribution for $\eta = \frac{1}{32}$, whereas Fig. 12 (right) shows an approximation of the second eigenfunction of (1.6)–(1.7) for a mesh of $32\times32\times32$ elements. The approximated Poincaré constants for a fixed mesh and varying η are displayed in Table 2. They suggest a behaviour of $C_{\mathrm{P},\alpha}(D) = \mathcal{O}(1+\log(H/\eta))$.

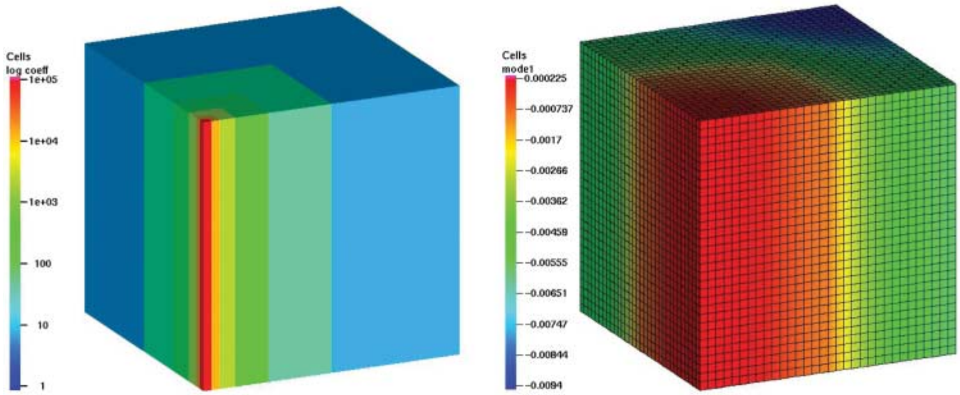


FIG. 12. Coefficient distribution and second eigenfunction for Example 4.6 for $\eta = \frac{1}{32}$ and $h = \frac{1}{32}$.

TABLE 2 *Approximate Poincaré constants for Example* 4.6 *for the fixed mesh parameter* $h = \frac{1}{32}$

η	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$C_{\mathrm{P},\alpha}(D)$	0.0588303	0.0637642	0.0700526	0.0764003

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Appendix

LEMMA A1 Let K be a (nondegenerate) d-dimensional simplex ($d \geqslant 2$) and let F be one of its facets. Then

$$C_P(K; F) \leqslant 1.$$

If K is a d-dimensional parallelepiped then $C_P(K; F) \leqslant \frac{7}{5}$.

Proof. Veeser & Verfürth (2009, Section 4, Remark 4.6, Formula (2.3) and Corollary 4.5) have shown that, $\forall\, v \in H^1(K)$,

$$\frac{1}{\mathrm{meas}_{d-1}(F)} \|v\|^2_{L^2(F)} \leqslant \frac{1}{\mathrm{meas}_d(K)} \|v\|^2_{L^2(K)} + \frac{2\,\mathrm{diam}(K)}{\nu_K\, \mathrm{meas}_d(K)} \|v\|_{L^2(K)} |v|_{H^1(K)}, \tag{A.1}$$

where $\nu_K = d$ for the simplex and $\nu_K = 1$ for the parallelepiped. Owing to Payne & Weinberger (1960) and Bebendorf (2003),

$$\|u - \bar{u}^K\|_{L^2(K)} \leqslant \frac{\mathrm{diam}(K)}{\pi} |u|_{H^1(K)} \quad \forall u \in H^1(K), \tag{A.2}$$

because K is convex. With the triangle inequality and Cauchy's inequality,

$$\begin{aligned}
\|u - \bar{u}^F\|_{L^2(K)} &\leqslant \|u - \bar{u}^K\|_{L^2(K)} + \sqrt{\mathrm{meas}_d(K)}|\bar{u}^K - \bar{u}^F| \\
&\leqslant \|u - \bar{u}^K\|_{L^2(K)} + \frac{\sqrt{\mathrm{meas}_d(K)}}{\sqrt{\mathrm{meas}_{d-1}(F)}} \|u - \bar{u}^K\|_{L^2(F)}.
\end{aligned}$$

Using (A.1) and (A.2) in the estimate above yields

$$\begin{aligned}
\|u - \bar{u}^F\|_{L^2(K)} &\leqslant \frac{\mathrm{diam}(K)}{\pi} |u|_{H^1(K)} + \sqrt{\|u - \bar{u}^K\|^2_{L^2(K)} + \frac{2\,\mathrm{diam}(K)}{\nu_K} \|u - \bar{u}^K\|_{L^2(K)} |u|_{H^1(K)}} \\
&\leqslant \frac{\mathrm{diam}(K)}{\pi} |u|_{H^1(K)} + \sqrt{\frac{\mathrm{diam}(K)^2}{\pi^2} |u|^2_{H^1(K)} + \frac{2\,\mathrm{diam}(K)}{\nu_K} \frac{\mathrm{diam}(K)}{\pi} |u|^2_{H^1(K)}} \\
&= \underbrace{\left(\frac{1}{\pi} + \sqrt{\frac{1}{\pi^2} + \frac{2}{\nu_K \pi}} \right)}_{:=C} \mathrm{diam}(K) |u|_{H^1(K)}.
\end{aligned}$$

For the simplex, $\nu_K = d \geqslant 2$ and we get $C \leqslant 0.96609936 < 1$. For the parallelepiped, $\nu_K = 1$ and so $C \leqslant 1.17734478 < \sqrt{\frac{7}{5}}$. □

LEMMA A2 Let T be a (nondegenerate) d-dimensional simplex and let $\rho(T)$ be the diameter of the largest inscribed ball in $\bar{T}$. Then

$$\operatorname{meas}_d(T) \geqslant \operatorname{diam}(T)\left(\frac{\rho(T)}{2}\right)^{d-1}. \tag{A.3}$$

Proof. The proof is by induction. For $d = 1$, the inequality is trivial since $\rho(T) = \operatorname{diam}(T) = \operatorname{meas}_1(T)$.

Let $d > 1$ and assume that (A.3) holds for $d - 1$. Let F_i, $i = 1, \ldots, d + 1$ be the $(d - 1)$-dimensional facets of T. Applying the induction hypothesis to each F_i, we obtain

$$\begin{aligned}\operatorname{meas}_{d-1}(\partial T) = \sum_{i=1}^{d+1} \operatorname{meas}_{d-1}(F_i) &\geqslant \sum_{i=1}^{d+1} \operatorname{diam}(F_i)\left(\frac{\rho(F_i)}{2}\right)^{d-2} \\ &\geqslant \left(\frac{\rho(T)}{2}\right)^{d-2} \sum_{i=1}^{d+1} \operatorname{diam}(F_i),\end{aligned}$$

where in the last step we used that $\rho(T) \leqslant \rho(F_i)$.

Let E be the longest edge of T such that $\operatorname{diam}(T) = \operatorname{diam}(E)$. Since each of the facets F_i contains d vertices, there are exactly two facets F_{k_1}, F_{k_2} that do not contain E. The remaining facets contain E and hence their diameters equal $\operatorname{diam}(E) = \operatorname{diam}(T)$. The facets F_{k_1}, F_{k_2} share $(d - 1)$ vertices, so we can find edges $E_{k_1} \subset F_{k_1}$ and $E_{k_1} \subset F_{k_2}$ that meet at such a vertex, and such that $[E, E_{k_1}, E_{k_2}]$ is a proper triangle. From this, it is easily seen that $\operatorname{diam}(F_{k_1}) + \operatorname{diam}(F_{k_2}) \geqslant \operatorname{diam}(E) = \operatorname{diam}(T)$, and so

$$\sum_{i=1}^{d+1} \operatorname{diam}(F_i) \geqslant d \operatorname{diam}(T).$$

We split T into $d + 1$ simplices T_i, $i = 1, \ldots, d + 1$, defined such that T_i has F_i as its base and the centre of the largest inscribed ball as its apex. Obviously, the height of T_i is $\rho(T)/2$ and, hence, $\operatorname{meas}_d(T_i) = (1/d) \operatorname{meas}_{d-1}(F_i)(\rho(T)/2)$. Summing over $i = 1, \ldots, d + 1$, we get

$$\operatorname{meas}_d(T) = \frac{1}{d}\frac{\rho(T)}{2} \operatorname{meas}_{d-1}(\partial T).$$

Using this identity and the inequalities from above, we obtain

$$\operatorname{meas}_d(T) \geqslant \frac{1}{d}\frac{\rho(T)}{2}\left(\frac{\rho(T)}{2}\right)^{d-2} d \operatorname{diam}(T) = \operatorname{diam}(T)\left(\frac{\rho(T)}{2}\right)^{d-1},$$

that is, inequality (A.3) holds for d. □

Proof of Lemma 3.1. First, we need to provide three auxiliary lemmas.

LEMMA A3 Let Q be a Lipschitz polytope and $V^h(Q)$ be the FE space of continuous, piecewise d-linear FE functions corresponding to a family of quasi-uniform meshes. Then there exists a positive constant C independent of h (but depending on Q and d) such that

$$\|u\|^2_{L^\infty(Q)} \leqslant C\sigma^d(h^{-1})\|u\|^2_{H^1(Q)} \quad \forall u \in V^h(Q).$$

Proof. For $d = 1$ the estimate follows from the trace theorem; for $d = 2$, see, for example, Toselli & Widlund (2005, Lemma 4.15). For $d \geqslant 3$, the space $H^1(Q)$ is embedded in $L^q(Q)$ with $q = 2d/(d-2)$. By an inverse inequality, we obtain

$$\|u\|_{L^\infty(Q)} \lesssim h^{-d/q} \|u\|_{L^q(Q)} \lesssim h^{-(d-2)/2} \|u\|_{H^1(Q)}.$$

□

Lemma A4 Let Y be the d-dimensional hypercube $(0,1)^d$, let X be one of its m-facets and let $V^h(Y)$ be the space of continuous, piecewise d-linear FE functions corresponding to a uniform Cartesian mesh. Then there exists a positive constant C independent of h such that

$$\|u\|^2_{L^2(X)} \leqslant C\sigma^{d-m}(h^{-1}) \|u\|^2_{H^1(Y)} \quad \forall u \in V^h(Y).$$

Proof. Let Q be the $(d-m)$-dimensional hypercube such that $Y = X \times Q$. Since the mesh is Cartesian, its intersection with $\{x\} \times Q$ is again a uniform Cartesian mesh. If we write (x, q) for the coordinates in $X \times Q$, we can conclude that, for any $x \in X$, $u(x, \cdot) \in V^h(Q)$. Thus, with Lemma A3,

$$\begin{aligned} \|u\|^2_{L^2(X)} &= \int_X |u(x,0)|^2 \, \mathrm{d}x \leqslant \int_X \|u(x,\cdot)\|^2_{L^\infty(Q)} \, \mathrm{d}x \\ &\lesssim \sigma^{d-m}(h^{-1}) \int_X \|u(x,\cdot)\|^2_{H^1(Q)} \, \mathrm{d}x \lesssim \sigma^{d-m}(h^{-1}) \|u\|^2_{H^1(Y)}. \end{aligned}$$

□

Lemma A5 The statement of Lemma A4 holds also for general quasi-uniform meshes.

Proof. For a fixed mesh $\mathcal{T}^h(Y) = \{T\}$, we can always find a uniform Cartesian mesh $\mathcal{Q}^h(Y) = \{K\}$ with comparable mesh size, such that, for each $T \in \mathcal{T}^h(Y)$, there is an element $K \in \mathcal{Q}^h(Y)$ such that $K \subset T$. This restriction guarantees that

$$\|u\|_{L^2(X)} \lesssim \|I_{\mathcal{Q}} u\|_{L^2(X)}, \quad \|I_{\mathcal{Q}} u\|_{H^1(Y)} \lesssim \|u\|_{H^1(Y)} \ \ \forall u \in V^h(Y),$$

where $I_{\mathcal{Q}}$ is the nodal interpolator onto $\mathcal{Q}^h$; see Toselli & Widlund (2005, Lemmas 3.8 and B.5). With these considerations, the result follows simply from Lemma A4 applied to $I_{\mathcal{Q}} u$. The additional factors introduced by $I_{\mathcal{Q}}$ only depend on the shape-regularity constant of $\mathcal{T}^h(Y)$. □

We can now prove Lemma 3.1. Without loss of generality, we may assume that $\mathrm{diam}(Y) = 1$, and the general case follows from a simple scaling argument. We can find auxiliary domains $Y_i \subset Y$, $i = 1, \ldots, m+1$ such that (a) $\bigcup_{i=1}^{m+1} \partial Y_i \cap X = X$ and (b) each domain Y_i is the image of the m-dimensional hypercube under a d-linear map whose Jacobian (and its inverse) can be uniformly bounded in terms of $\rho(Y)$. We may assume that the mesh resolves the domains Y_i; otherwise we can find an auxiliary mesh with this property and interpolate between the two meshes as in the proof above. Applying Lemma A5 to each Y_i and summing over $i = 1, \ldots, m+1$, we get

$$\|u\|_{L^2(X)} \lesssim \sigma^{d-m}(h^{-1})\|u\|_{H^1(Y)} \quad \forall u \in V^h(Y).$$

Finally, the result of Lemma 3.1 can be proved by a classical Bramble–Hilbert argument. By Cauchy's inequality, the standard Poincaré inequality and the result above, we have

$$\begin{aligned}\|u - \bar{u}^X\|^2_{L^2(Y)} &\lesssim \|u - \bar{u}^Y\|^2_{L^2(Y)} + \frac{\mathrm{meas}_d(Y)}{\mathrm{meas}_m(X)}\|u - \bar{u}^Y\|^2_{L^2(X)} \\ &\lesssim [C_P(Y) + \sigma^{d-m}(h^{-1})C_P(Y)]|u|^2_{H^1(Y)}.\end{aligned}$$

□